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Research Paper

A Note on Generalized Bell-Appell Polynomials

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Abstract

This paper aims to establish a new hybrid class of special polynomials, namely, the generalized Bell-Appell polynomials. The idea of the monomiality principle is used to construct the generating function for the generalized Bell-Appell polynomials. Certain related identities and properties are also considered. The determinant representation is also derived. Further, we present some special cases of generalized Bell-Appell family and investigate the corresponding results.

Key Words: Bell Polynomials, Generalized Bell Polynomials, Appell Polynomials, Bernoulli Polynomials, Euler Polynomials, Generating Function, Determinant Representation

AMS 2020 Classification: 11B73, 11B83, 05A15, 33C47

1. Introduction

Special polynomials have significant roles in many branches of mathematics, theoretical physics, and engineering [1, 2, 3]. We realize that various problems in engineering and physics are framed in terms of differential equations, and most of these equations can be investigated by using several families of special polynomials. Further, these special polynomials allow the derivation of various helpful identities in a fairly straight forward way and useful in introducing new classes of special polynomials. The Bell polynomials are of the most important special polynomials due to their various applications in different mathematical frameworks (see [1, 3, 4]). The Appell polynomials arise in various applications in pure and applied mathematics. These interesting polynomials appear in chemistry, theoretical physics and many other branches of mathematics such as the study of polynomial expansions of analytic functions, numerical analysis, and number theory (see [5, 6, 7]). Throughout this study, the following notations and definitions are used: $\mathbb{N} = \{1, 2, 3, ...\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

The 2-variable Bell polynomials (2VBelP) $\mathcal{B}el_\varepsilon(\upsilon_1, \upsilon_2)$ [8, 9] are defined by

$$\exp\left(\upsilon_1\omega + \upsilon_2(e^{\omega} - 1)\right) = \sum_{\varepsilon=0}^{\infty} \mathcal{B}el_\varepsilon(\upsilon_1, \upsilon_2)\ \frac{\omega^{\varepsilon}}{\varepsilon!}. \tag{1}$$

Taking $\upsilon_1 = 0$ in generating function (1), we get

$$\exp(\upsilon_2(e^{\omega}-1))=\sum_{\varepsilon=0}^{\infty}\mathcal{B}el_{\varepsilon}(\upsilon_2)\ \frac{\omega^{\varepsilon}}{\varepsilon!},\tag{2}$$

where $\mathcal{B}el_{\varepsilon}(\upsilon_2)$ denotes the classical Bell polynomials [2, 10, 11].
The generalized Bell polynomials (GBP) ${}_{\mathcal{G}}\mathcal{B}el_{\varepsilon}(\upsilon_1,\upsilon_2,z)$ [12] are defined by

$$\exp(\upsilon_1\omega)\psi(\upsilon_2,\omega)\ \exp(z(e^{\omega}-1))=\sum_{\varepsilon=0}^{\infty}{}_{\mathcal{G}}\mathcal{B}el_{\varepsilon}(\upsilon_1,\upsilon_2,z)\ \frac{\omega^{\varepsilon}}{\varepsilon!}.\tag{3}$$

Setting $\upsilon_1=0$ in generating relation (3), we get

$$\psi(\upsilon_2,\omega)\ e^{z(e^{\omega}-1)}=\sum_{\varepsilon=0}^{\infty}{}_{\mathcal{G}}\mathcal{B}el_{\varepsilon}(\upsilon_2,z)\ \frac{\omega^{\varepsilon}}{\varepsilon!},\tag{4}$$

where ${}_{\mathcal{G}}\mathcal{B}el_{\varepsilon}(\upsilon_2,z)$ are called 2-variable generalized Bell polynomials.
The generalized Bell polynomials ${}_{\mathcal{G}}\mathcal{B}el_{\varepsilon}(\upsilon_1,\upsilon_2,z)$ satisfy the following series representations:

$${}_{\mathcal{G}}\mathcal{B}el_{\varepsilon}(\upsilon_1,\upsilon_2,z)=\sum_{\kappa=0}^{\varepsilon}\binom{\varepsilon}{\kappa}\mathcal{G}_{\varepsilon-\kappa}(\upsilon_1,\upsilon_2)\ \mathcal{B}el_{\kappa}(z);\tag{5}$$

$${}_{\mathcal{G}}\mathcal{B}el_{\varepsilon}(\upsilon_1,\upsilon_2,z)=\sum_{\kappa=0}^{\varepsilon}\binom{\varepsilon}{\kappa}\mathcal{B}el_{\varepsilon-\kappa}(\upsilon_1,z)\ \psi_{\kappa}(\upsilon_2);\tag{6}$$

$${}_{\mathcal{G}}\mathcal{B}el_{\varepsilon}(\upsilon_1,\upsilon_2,z)=\sum_{\kappa=0}^{\varepsilon}\binom{\varepsilon}{\kappa}{}_{\mathcal{G}}\mathcal{B}el_{\kappa}(\upsilon_2,z)\ \upsilon_1^{\varepsilon-\kappa}.\tag{7}$$

The generalized Bell family contains many important polynomials. We present the list of some known generalized Bell family in Table 1.

S.No.	$\psi(\upsilon_2,\omega)$	Polynomial	Generating Function
I.	$\exp(\upsilon_2\omega^r)$	Gould-Hopper-Bell polynomials [12]	$\exp(\upsilon_1 t+\upsilon_2\omega^r+z(e^{\omega}-1))$ $=\sum_{\varepsilon=0}^{\infty}{}_{\mathcal{H}^{(r)}}\mathcal{B}el_{\varepsilon}(\upsilon_1,\upsilon_2,z)\frac{\omega^{\varepsilon}}{\varepsilon!}$
II.	$C_0(\upsilon_2\omega)$	Laguerre-Bell polynomials [12]	$C_0(\upsilon_2\omega)\ \exp(\upsilon_1\omega+z(e^{\omega}-1))$ $=\sum_{\varepsilon=0}^{\infty}{}_{L}\mathcal{B}el_{\varepsilon}(\upsilon_1,\upsilon_2,z)\frac{\omega^{\varepsilon}}{\varepsilon!}$
III.	$\frac{1}{1-\upsilon_2\omega^s}$	truncated-exponential-Bell polynomials of order s [12]	$\frac{1}{1-\upsilon_2\omega^s}\ \exp(\upsilon_1\omega+z(e^{\omega}-1))$ $=\sum_{\varepsilon=0}^{\infty}{}_{e^{(s)}}\mathcal{B}el_{\varepsilon}(\upsilon_1,\upsilon_2,z)\ \frac{\omega^{\varepsilon}}{\varepsilon!}$
IV.	$\frac{1}{1-\upsilon_2(e^{\omega}-1)}$	Fubini-Bell polynomials [12]	$\frac{\exp(\upsilon_1\omega)}{1-\upsilon_2(e^{\omega}-1)}\ \exp(z(e^{\omega}-1))$ $=\sum_{\varepsilon=0}^{\infty}{}_{\mathcal{F}}\mathcal{B}el_{\varepsilon}(\upsilon_1,\upsilon_2,z)\ \frac{\omega^{\varepsilon}}{\varepsilon!}$

Table 1. Certain members belonging to the generalized Bell polynomials ${}_{\mathcal{G}}\mathcal{B}el_{\varepsilon}(\upsilon_1,\upsilon_2,z)$.

The generalized Bell polynomials ${}_{\mathcal{G}}\mathcal{B}el_{\varepsilon}(\upsilon_1,\upsilon_2,z)$ are quasi-monomial with respect to the following multiplicative and derivative operators [12]:

$$\hat{M}_{\mathcal{G}\mathcal{B}el}=\upsilon_1+\frac{\psi'(\upsilon_2,D_{\upsilon_1})}{\psi(\upsilon_2,D_{\upsilon_1})}+ze^{D_{\upsilon_1}}\tag{8}$$

and

$$\hat{P}_{\mathcal{G}\mathcal{B}el}=D_{\upsilon_1},\tag{9}$$

respectively.

According to the monomiality principle, the GBP ${}_{\mathcal{G}}\mathcal{B}el_{\varepsilon}(\upsilon_1,\upsilon_2,z)$ satisfy the following identities:

$$\hat{M}_{\mathcal{G}\mathcal{B}el}\{{}_{\mathcal{G}}\mathcal{B}el_{\varepsilon}(\upsilon_1,\upsilon_2,z)\} = {}_{\mathcal{G}}\mathcal{B}el_{\varepsilon+1}(\upsilon_1,\upsilon_2,z), \tag{10}$$

$$\hat{P}_{\mathcal{G}\mathcal{B}el}\{{}_{\mathcal{G}}\mathcal{B}el_{\varepsilon}(\upsilon_1,\upsilon_2,z)\} = \varepsilon\ {}_{\mathcal{G}}\mathcal{B}el_{\varepsilon-1}(\upsilon_1,\upsilon_2,z), \tag{11}$$

$$\hat{M}_{\mathcal{G}\mathcal{B}el}\hat{P}_{\mathcal{G}\mathcal{B}el}\{{}_{\mathcal{G}}\mathcal{B}el_{\varepsilon}(\upsilon_1,\upsilon_2,z)\} = \varepsilon\ {}_{\mathcal{G}}\mathcal{B}el_{\varepsilon}(\upsilon_1,\upsilon_2,z), \tag{12}$$

$$\exp(\hat{M}_{\mathcal{G}\mathcal{B}el}\omega)\{1\} = \sum_{\varepsilon=0}^{\infty} {}_{\mathcal{G}}\mathcal{B}el_{\varepsilon}(\upsilon_1,\upsilon_2,z)\ \frac{\omega^{\varepsilon}}{\varepsilon!} \qquad (|\omega|<\infty). \tag{13}$$

The sequences of Appell polynomial (AP) surface in various applicable problems in Applied and pure mathematics such as the investigation and study of analytic problems, polynomial expansions in physics and chemistry[5, 6, 7].

The Appell polynomial sets [13] might be characterized by either of the equivalent conditions [14, p.398]: $\{\mathcal{A}_{\varepsilon}(\upsilon_1)\}(\varepsilon = 0,1,2,3,...)$, is an Appell set ($\mathcal{A}_{\varepsilon}(\upsilon_1)$ of degree exactly ε) if either

(a) $\dfrac{d}{d\upsilon_1}\mathcal{A}_{\varepsilon}(\upsilon_1) = \varepsilon\ \mathcal{A}_{\varepsilon-1}(\upsilon_1), \quad \varepsilon = 0,1,2,3,...,$ or

(b) there exists formal series

$$\mathcal{A}(\omega) = \sum_{\varepsilon=0}^{\infty} \mathcal{A}_{\varepsilon}\ \frac{\omega^{\varepsilon}}{\varepsilon!}, \qquad\qquad \mathcal{A}_0 \neq 0, \tag{14}$$

such that (again formally)

$$\mathcal{A}(\omega)\ \exp(\upsilon_1\omega) = \sum_{\varepsilon=0}^{\infty} \mathcal{A}_{\varepsilon}(\upsilon_1)\ \frac{\omega^{\varepsilon}}{\varepsilon!}. \tag{15}$$

Özat et al [15] defined Bell based Appell polynomials as

$$\mathcal{A}(\omega)\quad \exp(\upsilon_1\omega + \upsilon_2(e^{\omega}-1)) = \sum_{\varepsilon=0}^{\infty} {}_{\mathcal{B}el}\mathcal{A}_{\varepsilon}(\upsilon_1,\upsilon_2)\ \frac{\omega^{\varepsilon}}{\varepsilon!}. \tag{16}$$

Over the last few years, there has been increasing interest in a new approach related to special functions, that is, determinant approach. Costabile et al. [16] have established a new definition of Bernoulli polynomials based on a determinant approach. Further, Longo an Costabile have established determinant approaches to Sheffer and Appell polynomials (see [17, 18]). This stimulated the authors to shed light on the determinant approach of some new hybrid polynomials.

Recently, numerous researchers have utilized the operational methods together with the monomiality principle [19] to establish and investigate new mixed families of special polynomials [20, 21, 22, 23, 24, 25, 26, 27, 28].

In this work, by combining the generalized Bell polynomials and Appell polynomials, we present new family of hybrid special polynomials, namely the generalized Bell-Appell polynomials, that is in Definition 1. Next, the series representations and certain other important formulas for the generalized Bell-Appell polynomials are derived. In Section 3, we establish the determinant representation including these polynomials. Finally, certain special cases of the generalized Bell-Appell polynomials are discussed and the repleted results is obtained.

2. Generalized Bell-Appell Polynomials

Here, we introduce a new interesting class of hybrid special polynomials, called the generalized Bell-Appell polynomials by means of the generating functions.

In generating function (15), replacing υ_1 by the multiplicative operator $\hat{M}_{\mathcal{G}\mathcal{B}el}$ (8) of the GBP ${}_{\mathcal{G}}\mathcal{B}el_\varepsilon(\upsilon_1, \upsilon_2, z)$, gives

$$\mathcal{A}(\omega)\ \exp(\hat{M}_{\mathcal{G}\mathcal{B}el}\ \omega) = \sum_{\varepsilon=0}^{\infty} \mathcal{A}_\varepsilon(\hat{M}_{\mathcal{G}\mathcal{B}el})\ \frac{\omega^\varepsilon}{\varepsilon!}. \tag{17}$$

Using equation (13) in the above equation and denoting $\mathcal{A}_\varepsilon(\hat{M}_{\mathcal{G}\mathcal{B}el})$ by the resultant generalized Bell-Appell polynomials (GBAP) ${}_{\mathcal{G}\mathcal{B}el}\mathcal{A}_\varepsilon(\upsilon_1, \upsilon_2, z)$, gives

$$\mathcal{A}(\omega)\ \left(\sum_{\varepsilon=0}^{\infty} {}_{\mathcal{G}}\mathcal{B}el_\varepsilon(\upsilon_1, \upsilon_2, z)\ \frac{\omega^\varepsilon}{\varepsilon!}\right) = \sum_{\varepsilon=0}^{\infty} {}_{\mathcal{G}\mathcal{B}el}\mathcal{A}_\varepsilon(\upsilon_1, \upsilon_2, z)\ \frac{\omega^\varepsilon}{\varepsilon!}. \tag{18}$$

Now, utilizing equation (3) in the above equation, we arrive at the following definition.

Definition 1. The generalized Bell-Appell polynomials ${}_{\mathcal{G}\mathcal{B}el}\mathcal{A}_\varepsilon(\upsilon_1, \upsilon_2, z)$ are defined by the generating function:

$$\mathcal{A}(\omega)\ \psi(\upsilon_2, \omega)\ \exp(\upsilon_1\omega + z(e^\omega - 1)) = \sum_{\varepsilon=0}^{\infty} {}_{\mathcal{G}\mathcal{B}el}\mathcal{A}_\varepsilon(\upsilon_1, \upsilon_2, z)\ \frac{\omega^\varepsilon}{\varepsilon!}. \tag{19}$$

Remark 1. Setting $\upsilon_1 = 0$ in generating relation (19), we get two-variable generalized Bell-Appell polynomials ${}_{\mathcal{G}\mathcal{B}el}\mathcal{A}_\varepsilon(\upsilon_2, z)$ which are given as:

$$\mathcal{A}(\omega)\ \psi(\upsilon_2, \omega)\ \exp(z(e^\omega - 1)) = \sum_{\varepsilon=0}^{\infty} {}_{\mathcal{G}\mathcal{B}el}\mathcal{A}_\varepsilon(\upsilon_2, z)\ \frac{\omega^\varepsilon}{\varepsilon!}. \tag{20}$$

Remark 2. Setting $z = 0$ in generating relation (19), we get the 2-variable general-Appell polynomials ${}_{\mathcal{G}}\mathcal{A}_\varepsilon(\upsilon_1, \upsilon_2)$ [29] given by generating function

$$\mathcal{A}(\omega)\ \psi(\upsilon_2, \omega)\ \exp(\upsilon_1\omega) = \sum_{\varepsilon=0}^{\infty} {}_{\mathcal{G}}\mathcal{A}_\varepsilon(\upsilon_1, \upsilon_2)\ \frac{\omega^\varepsilon}{\varepsilon!}. \tag{21}$$

Next, by using generating function (19), we establish some novel identities and relations including the generalized Bell-Appell polynomials.

Theorem 1. *The generalized Bell-Appell polynomials* ${}_{\mathcal{G}\mathcal{B}el}\mathcal{A}_\varepsilon(\upsilon_1, \upsilon_2, z)$ *satisfy the following series representations:*

$${}_{\mathcal{G}\mathcal{B}el}\mathcal{A}_\varepsilon(\upsilon_1, \upsilon_2, z) = \sum_{\kappa=0}^{\varepsilon} \binom{\varepsilon}{\kappa} \mathcal{A}_{\varepsilon-\kappa}(\upsilon_1)\ {}_{\mathcal{G}}\mathcal{B}el_\kappa(\upsilon_2, z); \tag{22}$$

$${}_{\mathcal{G}\mathcal{B}el}\mathcal{A}_\varepsilon(\upsilon_1, \upsilon_2, z) = \sum_{\kappa=0}^{\varepsilon} \binom{\varepsilon}{\kappa} {}_{\mathcal{G}}\mathcal{A}_{\varepsilon-\kappa}(\upsilon_1, \upsilon_2)\ \mathcal{B}el_\kappa(z); \tag{23}$$

$${}_{\mathcal{G}\mathcal{B}el}\mathcal{A}_\varepsilon(\upsilon_1, \upsilon_2, z) = \sum_{\kappa=0}^{\varepsilon} \binom{\varepsilon}{\kappa} {}_{\mathcal{G}\mathcal{B}el}\mathcal{A}_\kappa(\upsilon_2, z)\ \upsilon_1^{\varepsilon-\kappa}; \tag{24}$$

$${}_{\mathcal{G}\mathcal{B}el}\mathcal{A}_\varepsilon(\upsilon_1, \upsilon_2, z) = \sum_{\kappa=0}^{\varepsilon} \binom{\varepsilon}{\kappa} \mathcal{G}_{\varepsilon-\kappa}(\upsilon_1, \upsilon_2)\ {}_{\mathcal{B}el}\mathcal{A}_\kappa(z). \tag{25}$$

Proof. In view of generating relations (4) and (15) and Cauchy product rule, generating relation (19) gives

$$\sum_{\varepsilon=0}^{\infty} {}_{\mathcal{G}\mathcal{B}el}\mathcal{A}_\varepsilon(\upsilon_1, \upsilon_2, z)\ \frac{\omega^\varepsilon}{\varepsilon!} = \sum_{\varepsilon=0}^{\infty}\sum_{\kappa=0}^{\varepsilon} \binom{\varepsilon}{\kappa} \mathcal{A}_{\varepsilon-\kappa}(\upsilon_1)\ {}_{\mathcal{G}}\mathcal{B}el_\kappa(\upsilon_2, z)\ \frac{\omega^\varepsilon}{\varepsilon!}, \tag{26}$$

which, upon equating the coefficients of the analogous powers of ω, yields the assertion in equation (22). Similarly, the assertions in equations (23), (24) and (25) can be proved. □

Theorem 2. *For $\varepsilon \in \mathbb{N}_0$, we have*

$$ {}_{\mathcal{G}\mathcal{B}el}\mathcal{A}_{\varepsilon}(\upsilon_1, \upsilon_2, z) = \frac{1}{2}\sum_{\kappa=0}^{\varepsilon}\binom{\varepsilon}{\kappa}\mathcal{E}_{\kappa}\Big({}_{\mathcal{G}\mathcal{B}el}\mathcal{A}_{\varepsilon-\kappa}(\upsilon_1+1, \upsilon_2, z) + {}_{\mathcal{G}\mathcal{B}el}\mathcal{A}_{\varepsilon-\kappa}(\upsilon_1, \upsilon_2, z)\Big). \tag{27} $$

Proof. According generating relation (19), we can write

$$ \sum_{\varepsilon=0}^{\infty} {}_{\mathcal{G}\mathcal{B}el}\mathcal{A}_{\varepsilon}(\upsilon_1+1, \upsilon_2, z)\frac{\omega^{\varepsilon}}{\varepsilon!} + \sum_{\varepsilon=0}^{\infty} {}_{\mathcal{G}\mathcal{B}el}\mathcal{A}_{\varepsilon}(\upsilon_1, \upsilon_2, z)\frac{\omega^{\varepsilon}}{\varepsilon!} = (e^{\omega}+1)\sum_{\varepsilon=0}^{\infty} {}_{\mathcal{G}\mathcal{B}el}\mathcal{A}_{\varepsilon}(\upsilon_1, \upsilon_2, z)\frac{\omega^{\varepsilon}}{\varepsilon!}, \tag{28} $$

which can be written as

$$ \sum_{\varepsilon=0}^{\infty} {}_{\mathcal{G}\mathcal{B}el}\mathcal{A}_{\varepsilon}(\upsilon_1, \upsilon_2, z)\frac{\omega^{\varepsilon}}{\varepsilon!} = \frac{1}{2}\Big(\sum_{\varepsilon=0}^{\infty}\mathcal{E}_{\varepsilon}\frac{\omega^{\varepsilon}}{\varepsilon!}\Big)\Big(\sum_{\varepsilon=0}^{\infty} {}_{\mathcal{G}\mathcal{B}el}\mathcal{A}_{\varepsilon}(\upsilon_1+1, \upsilon_2, z)\frac{\omega^{\varepsilon}}{\varepsilon!} + \sum_{\varepsilon=0}^{\infty} {}_{\mathcal{G}\mathcal{B}el}\mathcal{A}_{\varepsilon}(\upsilon_1, \upsilon_2, z)\frac{\omega^{\varepsilon}}{\varepsilon!}\Big), \tag{29} $$

where $\mathcal{E}_{\varepsilon}$ denotes the Euler numbers [30]. Finally, using Cauchy product rule and comparing the like powers of ω in the resultant equation, we get (27). □

3. Determinant Representation

Theorem 3. *The generalized Bell-Appell polynomials ${}_{\mathcal{G}\mathcal{B}el}\mathcal{A}_{\varepsilon}(\upsilon_1, \upsilon_2, z)$ of degree ε are defined by*

$$ {}_{\mathcal{G}\mathcal{B}el}\mathcal{A}_0(x, y, z) = \frac{1}{\beta_0}, \qquad \beta_0 = \frac{1}{A_0}, \tag{30} $$

$$ {}_{\mathcal{G}\mathcal{B}el}\mathcal{A}_{\varepsilon}(\upsilon_1, \upsilon_2, z) = \frac{(-1)^{\varepsilon}}{(\beta_0)^{\varepsilon+1}}\begin{vmatrix} 1 & {}_{\mathcal{G}}\mathcal{B}el_1(\upsilon_1,\upsilon_2,z) & {}_{\mathcal{G}}\mathcal{B}el_2(\upsilon_1,\upsilon_2,z) & \dots & {}_{\mathcal{G}}\mathcal{B}el_{\varepsilon-1}(\upsilon_1,\upsilon_2,z) & {}_{\mathcal{G}}\mathcal{B}el_{\varepsilon}(\upsilon_1,\upsilon_2,z) \\ \beta_0 & \beta_1 & \beta_2 & \dots & \beta_{\varepsilon-1} & \beta_{\varepsilon} \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \dots & \binom{\varepsilon-1}{1}\beta_{\varepsilon-2} & \binom{n}{1}\beta_{\varepsilon-1} \\ 0 & 0 & \beta_0 & \dots & \binom{\varepsilon-1}{2}\beta_{\varepsilon-3} & \binom{\varepsilon}{2}\beta_{\varepsilon-2} \\ . & . & . & \dots & . & . \\ 0 & 0 & 0 & \dots & \beta_0 & \binom{\varepsilon}{\varepsilon-1}\beta_1 \end{vmatrix}, \tag{31} $$

$$ \beta_{\varepsilon} = -\frac{1}{\mathcal{A}_0}\Big(\sum_{k=1}^{\varepsilon}\binom{\varepsilon}{k}A_k\beta_{\varepsilon-k}\Big), \qquad \varepsilon = 1, 2, \dots, $$

where $\beta_0, \beta_1, \dots, \beta_{\varepsilon} \in \mathbb{R}, \beta_0 \neq 0$ and ${}_{\mathcal{G}}\mathcal{B}el_{\varepsilon}(\upsilon_1, \upsilon_2, z)(\varepsilon = 0, 1, 2, \dots,)$ are the generalized Bell polynomials defined by equation (3).

Proof. We start with the determinant definition of the AP $\mathcal{A}_{\varepsilon}(\upsilon_1)$ of degree ε which is given as [17]:

$$ \mathcal{A}_0(\upsilon_1) = \frac{1}{\beta_0}, \qquad \beta_0 = \frac{1}{\mathcal{A}_0}, \tag{32} $$

$$ \mathcal{A}_{\varepsilon}(\upsilon_1) = \frac{(-1)^{\varepsilon}}{(\beta_0)^{\varepsilon+1}}\begin{vmatrix} 1 & \upsilon_1 & \upsilon_1^2 & \dots & \upsilon_1^{\varepsilon-1} & \upsilon_1^{\varepsilon} \\ \beta_0 & \beta_1 & \beta_2 & \dots & \beta_{\varepsilon-1} & \beta_{\varepsilon} \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \dots & \binom{\varepsilon-1}{1}\beta_{\varepsilon-2} & \binom{\varepsilon}{1}\beta_{\varepsilon-1} \\ 0 & 0 & \beta_0 & \dots & \binom{\varepsilon-1}{2}\beta_{\varepsilon-3} & \binom{\varepsilon}{2}\beta_{\varepsilon-2} \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ 0 & 0 & 0 & \dots & \beta_0 & \binom{\varepsilon}{\varepsilon-1}\beta_1 \end{vmatrix}, \quad \beta_{\varepsilon} = -\frac{1}{\mathcal{A}_0}\Big(\sum_{k=1}^{\varepsilon}\binom{\varepsilon}{k}\mathcal{A}_k\beta_{\varepsilon-k}\Big), $$

$$\varepsilon = 1, 2, 3, ..., \tag{33}$$

where $\beta_0, \beta_1, ..., \beta_\varepsilon \in \mathbb{R}, \beta_0 \neq 0$.

Setting $\varepsilon = 0$ in series definition (22) and then using equation (32) in the resultant equation, we obtain assertion (30).

In order to prove assertion (31), the determinant of the Appell polynomials given in equation (33) is expanded with respect to the first row, so that

$$\mathcal{A}_\varepsilon(\upsilon_1) = \frac{(-1)^\varepsilon}{(\beta_0)^{\varepsilon+1}} \begin{vmatrix} \beta_1 & \beta_2 & ... & \beta_{\varepsilon-1} & \beta_\varepsilon \\ \beta_0 & \binom{2}{1}\beta_1 & ... & \binom{\varepsilon-1}{1}\beta_{\varepsilon-2} & \binom{\varepsilon}{1}\beta_{n-1} \\ 0 & \beta_0 & ... & \binom{\varepsilon-1}{2}\beta_{\varepsilon-3} & \binom{\varepsilon}{2}\beta_{\varepsilon-2} \\ . & . & ... & . & . \\ 0 & 0 & ... & \beta_0 & \binom{\varepsilon}{\varepsilon-1}\beta_1 \end{vmatrix}$$

$$-\frac{(-1)^\varepsilon \upsilon_1}{(\beta_0)^{\varepsilon+1}} \begin{vmatrix} \beta_0 & \beta_2 & ... & \beta_{\varepsilon-1} & \beta_\varepsilon \\ 0 & \binom{2}{1}\beta_1 & ... & \binom{\varepsilon-1}{1}\beta_{\varepsilon-2} & \binom{\varepsilon}{1}\beta_{\varepsilon-1} \\ 0 & \beta_0 & ... & \binom{\varepsilon-1}{2}\beta_{\varepsilon-3} & \binom{\varepsilon}{2}\beta_{\varepsilon-2} \\ . & . & ... & . & . \\ 0 & 0 & ... & \beta_0 & \binom{\varepsilon}{\varepsilon-1}\beta_1 \end{vmatrix} + \frac{(-1)^\varepsilon \upsilon_1^2}{(\beta_0)^{\varepsilon+1}} \begin{vmatrix} \beta_0 & \beta_1 & ... & \beta_{\varepsilon-1} & \beta_\varepsilon \\ 0 & \beta_0 & ... & \binom{\varepsilon-1}{1}\beta_{\varepsilon-2} & \binom{\varepsilon}{1}\beta_{\varepsilon-1} \\ 0 & 0 & ... & \binom{\varepsilon-1}{2}\beta_{\varepsilon-3} & \binom{\varepsilon}{2}\beta_{\varepsilon-2} \\ . & . & ... & . & . \\ 0 & 0 & ... & \beta_0 & \binom{\varepsilon}{\varepsilon-1}\beta_1 \end{vmatrix}$$

$$+ ... + \frac{(-1)^{2\varepsilon+1} \upsilon_1^{\varepsilon-1}}{(\beta_0)^{\varepsilon+1}} \begin{vmatrix} \beta_0 & \beta_1 & \beta_2 & ... & \beta_\varepsilon \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & ... & \binom{\varepsilon}{1}\beta_{\varepsilon-1} \\ 0 & 0 & \beta_0 & ... & \binom{\varepsilon}{2}\beta_{\varepsilon-2} \\ . & . & . & ... & . \\ 0 & 0 & 0 & ... & \binom{\varepsilon}{\varepsilon-1}\beta_1 \end{vmatrix} + \frac{\upsilon_1^\varepsilon}{(\beta_0)^{\varepsilon+1}} \begin{vmatrix} \beta_0 & \beta_1 & \beta_2 & ... & \beta_{\varepsilon-1} \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & ... & \binom{\varepsilon-1}{1}\beta_{\varepsilon-2} \\ 0 & 0 & \beta_0 & ... & \binom{\varepsilon-1}{2}\beta_{\varepsilon-3} \\ . & . & . & ... & . \\ 0 & 0 & 0 & ... & \beta_0 \end{vmatrix}. \tag{34}$$

Since each minor in equation (34) is independent of υ_1, therefore replacing υ_1 by the multiplicative operator $\hat{M}_{\mathcal{G}\mathcal{B}el}$ (8) in equation (34) and then using the monomiality principle equation ${}_{\mathcal{G}}\mathcal{B}el_\varepsilon(\upsilon_1, \upsilon_2, z) = \hat{M}^\varepsilon_{\mathcal{G}\mathcal{B}el}\{1\}(\varepsilon = 1, 2, ...)$, in the r.h.s. of the resultant equation, we find

$$\mathcal{A}_\varepsilon(\hat{M}_{\mathcal{G}\mathcal{B}el}) = \frac{(-1)^\varepsilon}{(\beta_0)^{\varepsilon+1}} \begin{vmatrix} \beta_1 & \beta_2 & ... & \beta_{\varepsilon-1} & \beta_\varepsilon \\ \beta_0 & \binom{2}{1}\beta_1 & ... & \binom{\varepsilon-1}{1}\beta_{\varepsilon-2} & \binom{\varepsilon}{1}\beta_{\varepsilon-1} \\ 0 & \beta_0 & ... & \binom{\varepsilon-1}{2}\beta_{\varepsilon-3} & \binom{\varepsilon}{2}\beta_{\varepsilon-2} \\ . & . & ... & . & . \\ 0 & 0 & ... & \beta_0 & \binom{\varepsilon}{\varepsilon-1}\beta_1 \end{vmatrix} - \frac{(-1)^\varepsilon {}_{\mathcal{G}}\mathcal{B}el_1(\upsilon_1, \upsilon_2, z)}{(\beta_0)^{\varepsilon+1}}$$

$$\times \begin{vmatrix} \beta_0 & \beta_2 & ... & \beta_{\varepsilon-1} & \beta_\varepsilon \\ 0 & \binom{2}{1}\beta_1 & ... & \binom{\varepsilon-1}{1}\beta_{\varepsilon-2} & \binom{\varepsilon}{1}\beta_{\varepsilon-1} \\ 0 & \beta_0 & ... & \binom{n-1}{2}\beta_{\varepsilon-3} & \binom{n}{2}\beta_{\varepsilon-2} \\ . & . & ... & . & . \\ 0 & 0 & ... & \beta_0 & \binom{\varepsilon}{\varepsilon-1}\beta_1 \end{vmatrix} + \frac{(-1)^\varepsilon {}_{\mathcal{G}}\mathcal{B}el_2(\upsilon_1, \upsilon_2, z)}{(\beta_0)^{\varepsilon+1}} \begin{vmatrix} \beta_0 & \beta_1 & ... & \beta_{\varepsilon-1} & \beta_\varepsilon \\ 0 & \beta_0 & ... & \binom{\varepsilon-1}{1}\beta_{\varepsilon-2} & \binom{\varepsilon}{1}\beta_{\varepsilon-1} \\ 0 & 0 & ... & \binom{\varepsilon-1}{2}\beta_{\varepsilon-3} & \binom{\varepsilon}{2}\beta_{\varepsilon-2} \\ . & . & ... & . & . \\ 0 & 0 & ... & \beta_0 & \binom{\varepsilon}{\varepsilon-1}\beta_1 \end{vmatrix}$$

$$+ ... + \frac{(-1)^{2\varepsilon+1} {}_{\mathcal{G}}\mathcal{B}el_{\varepsilon-1}(\upsilon_1, \upsilon_2, z)}{(\beta_0)^{\varepsilon+1}}$$

$$\times\begin{vmatrix}\beta_0 & \beta_1 & \beta_2 & \ldots & \beta_\varepsilon \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \ldots & \binom{\varepsilon}{1}\beta_{\varepsilon-1} \\ 0 & 0 & \beta_0 & \ldots & \binom{\varepsilon}{2}\beta_{\varepsilon-2} \\ . & . & . & \ldots & . \\ 0 & 0 & 0 & \ldots & \binom{\varepsilon}{\varepsilon-1}\beta_1\end{vmatrix} + \frac{{}_{\mathcal{G}}\mathcal{B}el_\varepsilon(\upsilon_1,\upsilon_2,z)}{(\beta_0)^{\varepsilon+1}}\begin{vmatrix}\beta_0 & \beta_1 & \beta_2 & \ldots & \beta_{\varepsilon-1} \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \ldots & \binom{\varepsilon-1}{1}\beta_{\varepsilon-2} \\ 0 & 0 & \beta_0 & \ldots & \binom{\varepsilon-1}{2}\beta_{\varepsilon-3} \\ . & . & . & \ldots & . \\ 0 & 0 & 0 & \ldots & \beta_0\end{vmatrix}. \tag{35}$$

Now, using the fact that $\mathcal{A}_\varepsilon(\hat{M}_{\mathcal{G}\mathcal{B}el}) = {}_{\mathcal{G}\mathcal{B}el}\mathcal{A}_\varepsilon(\upsilon_1,\upsilon_2,z)$ in the l.h.s. and combining the terms in the r.h.s. of equation (35), we get assertion (31). □

In the next section, certain special cases of the generalized Bell-Appell polynomials ${}_{\mathcal{G}\mathcal{B}el}\mathcal{A}_\varepsilon(\upsilon_1,\upsilon_2,z)$ are considered.

4. Special Cases

Here, we present new special hybrid members of the generalized Bell-Appell family ${}_{\mathcal{G}\mathcal{B}el}\mathcal{A}_\varepsilon(\upsilon_1,\upsilon_2,z)$. The obtained results in the previous sections are used to investigate the results that related to these new special hybrid members.

I. Taking $\mathcal{A}(\omega) = \frac{\omega}{e^\omega-1}$ in generating function (19), gives

$$\frac{\omega}{e^\omega-1}\ \psi(\upsilon_2,\omega)\ \exp(\upsilon_1\omega + z(e^\omega-1)) = \sum_{\varepsilon=0}^{\infty} {}_{\mathcal{G}\mathcal{B}el}\mathcal{B}_\varepsilon(\upsilon_1,\upsilon_2,z)\ \frac{\omega^\varepsilon}{\varepsilon!}, \tag{36}$$

where ${}_{\mathcal{G}\mathcal{B}el}\mathcal{B}_\varepsilon(\upsilon_1,\upsilon_2,z)$ is called the generalized Bell-Bernoulli polynomials (GBBP).

The GBBP ${}_{\mathcal{G}\mathcal{B}el}\mathcal{B}_\varepsilon(\upsilon_1,\upsilon_2,z)$ satisfy the following representations:

$${}_{\mathcal{G}\mathcal{B}el}\mathcal{B}_\varepsilon(\upsilon_1,\upsilon_2,z) = \sum_{\kappa=0}^{\varepsilon}\binom{\varepsilon}{\kappa}\mathcal{B}_{\varepsilon-\kappa}(\upsilon_1)\ {}_{\mathcal{G}}\mathcal{B}el_\kappa(\upsilon_2,z); \tag{37}$$

$${}_{\mathcal{G}\mathcal{B}el}\mathcal{B}_\varepsilon(\upsilon_1,\upsilon_2,z) = \sum_{\kappa=0}^{\varepsilon}\binom{\varepsilon}{\kappa}{}_{\mathcal{G}}\mathcal{B}_{\varepsilon-\kappa}(\upsilon_1,\upsilon_2)\ \mathcal{B}el_\kappa(z); \tag{38}$$

$${}_{\mathcal{G}\mathcal{B}el}\mathcal{B}_\varepsilon(\upsilon_1,\upsilon_2,z) = \sum_{\kappa=0}^{\varepsilon}\binom{\varepsilon}{\kappa}{}_{\mathcal{G}\mathcal{B}el}\mathcal{B}_\kappa(\upsilon_2,z)\ \upsilon_1^{\varepsilon-\kappa}; \tag{39}$$

$${}_{\mathcal{G}\mathcal{B}el}\mathcal{B}_\varepsilon(\upsilon_1,\upsilon_2,z) = \sum_{\kappa=0}^{\varepsilon}\binom{\varepsilon}{\kappa}\mathcal{G}_{\varepsilon-\kappa}(\upsilon_1,\upsilon_2)\ {}_{\mathcal{B}el}\mathcal{B}_\kappa(z); \tag{40}$$

$${}_{\mathcal{G}\mathcal{B}el}\mathcal{B}_\varepsilon(\upsilon_1,\upsilon_2,z) = \frac{1}{2}\sum_{\kappa=0}^{\varepsilon}\binom{\varepsilon}{\kappa}\ \mathcal{E}_\kappa\Big({}_{\mathcal{G}\mathcal{B}el}\mathcal{B}_{\varepsilon-\kappa}(\upsilon_1+1,\upsilon_2,z) + {}_{\mathcal{G}\mathcal{B}el}\mathcal{B}_{\varepsilon-\kappa}(\upsilon_1,\upsilon_2,z)\Big). \tag{41}$$

It has been shown in [17] that for $\beta_0 = 1$ and $\beta_j = \frac{1}{j+1}, (j = 1,2,3,...,\varepsilon)$ the determinant definition of Appell polynomials $\mathcal{A}_\varepsilon(\upsilon_1)$ defined by equations (32) and (33) reduces to determinant definition of Bernoulli polynomials $\mathcal{B}_\varepsilon(\upsilon_1)$ [16]. Therefore, taking $\beta_0 = 1$ and $\beta_j = \frac{1}{j+1}, (j = 1,2,3,...,\varepsilon)$ in equations (30) and (31), gives the following determinant form of the GBBP ${}_{\mathcal{G}\mathcal{B}el}\mathcal{B}_\varepsilon(\upsilon_1,\upsilon_2,z)$:

Corollary 1. The generalized Bell-Bernoulli polynomials ${}_{\mathcal{G}\mathcal{B}el}\mathcal{B}_\varepsilon(\upsilon_1,\upsilon_2,z)$ of degree ε are defined by

$${}_{\mathcal{G}\mathcal{B}el}\mathcal{B}_0(\upsilon_1,\upsilon_2,z) = 1, \tag{42}$$

$$ {}_{\mathcal{G}\mathcal{B}el}\mathcal{B}_\varepsilon(\upsilon_1,\upsilon_2,z)=(-1)^\varepsilon\begin{vmatrix} 1 & {}_{\mathcal{G}}\mathcal{B}el_1(\upsilon_1,\upsilon_2,z) & {}_{\mathcal{G}}\mathcal{B}el_2(\upsilon_1,\upsilon_2,z) & \ldots & {}_{\mathcal{G}}\mathcal{B}el_{\varepsilon-1}(\upsilon_1,\upsilon_2,z) & {}_{\mathcal{G}}\mathcal{B}el_\varepsilon(\upsilon_1,\upsilon_2,z) \\ 1 & \frac{1}{2} & \frac{1}{3} & \ldots & \frac{1}{\varepsilon} & \frac{1}{\varepsilon+1} \\ 0 & 1 & \binom{2}{1}\frac{1}{2} & \ldots & \binom{\varepsilon-1}{1}\frac{1}{\varepsilon-1} & \binom{\varepsilon}{1}\frac{1}{\varepsilon} \\ 0 & 0 & 1 & \ldots & \binom{\varepsilon-1}{2}\frac{1}{\varepsilon-2} & \binom{\varepsilon}{2}\frac{1}{\varepsilon-1} \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ 0 & 0 & 0 & \ldots & 1 & \binom{\varepsilon}{\varepsilon-1}\frac{1}{2} \end{vmatrix}, \tag{43} $$

$\varepsilon=1,2,3,...,$ where ${}_{\mathcal{G}}\mathcal{B}el_\varepsilon(\upsilon_1,\upsilon_2,z)(\varepsilon=0,1,2,...,)$ are the generalized Bell polynomials of degree ε.

II. Taking $\mathcal{A}(\omega)=\frac{2}{e^\omega+1}$ in generating function (19), gives

$$ \frac{2}{e^\omega+1}\ \psi(\upsilon_2,\omega)\ \exp(\upsilon_1\omega+z(e^\omega-1))=\sum_{\varepsilon=0}^{\infty}{}_{\mathcal{G}\mathcal{B}el}\mathcal{E}_\varepsilon(\upsilon_1,\upsilon_2,z)\ \frac{\omega^\varepsilon}{\varepsilon!}, \tag{44} $$

where ${}_{\mathcal{G}\mathcal{B}el}\mathcal{E}_\varepsilon(\upsilon_1,\upsilon_2,z)$ is called the generalized Bell-Euler polynomials (GBEP).

The GBEP ${}_{\mathcal{G}\mathcal{B}el}\mathcal{E}_\varepsilon(\upsilon_1,\upsilon_2,z)$ satisfy the following representations:

$$ {}_{\mathcal{G}\mathcal{B}el}\mathcal{E}_\varepsilon(\upsilon_1,\upsilon_2,z)=\sum_{\kappa=0}^{\varepsilon}\binom{\varepsilon}{\kappa}\mathcal{E}_{\varepsilon-\kappa}(\upsilon_1)\ {}_{\mathcal{G}}\mathcal{B}el_\kappa(\upsilon_2,z); \tag{45} $$

$$ {}_{\mathcal{G}\mathcal{B}el}\mathcal{E}_\varepsilon(\upsilon_1,\upsilon_2,z)=\sum_{\kappa=0}^{\varepsilon}\binom{\varepsilon}{\kappa}{}_{\mathcal{G}}\mathcal{E}_{\varepsilon-\kappa}(\upsilon_1,\upsilon_2)\ \mathcal{B}el_\kappa(z); \tag{46} $$

$$ {}_{\mathcal{G}\mathcal{B}el}\mathcal{E}_\varepsilon(\upsilon_1,\upsilon_2,z)=\sum_{\kappa=0}^{\varepsilon}\binom{\varepsilon}{\kappa}{}_{\mathcal{G}\mathcal{B}el}\mathcal{E}_\kappa(\upsilon_2,z)\ \upsilon_1^{\varepsilon-\kappa}; \tag{47} $$

$$ {}_{\mathcal{G}\mathcal{B}el}\mathcal{E}_\varepsilon(\upsilon_1,\upsilon_2,z)=\sum_{\kappa=0}^{\varepsilon}\binom{\varepsilon}{\kappa}\mathcal{G}_{\varepsilon-\kappa}(\upsilon_1,\upsilon_2)\ {}_{\mathcal{B}el}\mathcal{E}_\kappa(z); \tag{48} $$

$$ {}_{\mathcal{G}\mathcal{B}el}\mathcal{E}_\varepsilon(\upsilon_1,\upsilon_2,z)=\frac{1}{2}\sum_{\kappa=0}^{\varepsilon}\binom{\varepsilon}{\kappa}\ \mathcal{E}_\kappa\Big({}_{\mathcal{G}\mathcal{B}el}\mathcal{E}_{\varepsilon-\kappa}(\upsilon_1+1,\upsilon_2,z)+{}_{\mathcal{G}\mathcal{B}el}\mathcal{E}_{\varepsilon-\kappa}(\upsilon_1,\upsilon_2,z)\Big). \tag{49} $$

Further, according to the fact that for $\beta_0=1$ and $\beta_j=\frac{1}{2},(j=1,2,3,...,\varepsilon)$, equations (32) and (33) gives to the determinant form of Euler polynomials $\mathcal{E}_\varepsilon(\upsilon_1)$ [17], so by taking $\beta_0=1$ and $\beta_j=\frac{1}{2},(j=1,2,3,...,\varepsilon)$ in equations (30) and (31), gives the following determinant form of the GBEP ${}_{\mathcal{G}\mathcal{B}el}\mathcal{E}_\varepsilon(\upsilon_1,\upsilon_2,z)$:

Corollary 2. The generalized Bell-Euler polynomials ${}_{\mathcal{G}\mathcal{B}el}\mathcal{E}_\varepsilon(\upsilon_1,\upsilon_2,z)$ of degree ε are defined by

$$ {}_{\mathcal{G}\mathcal{B}el}\mathcal{E}_0(\upsilon_1,\upsilon_2,z)=1, \tag{50} $$

$$ {}_{\mathcal{G}\mathcal{B}el}\mathcal{E}_\varepsilon(\upsilon_1,\upsilon_2,z)=(-1)^\varepsilon\begin{vmatrix} 1 & {}_{\mathcal{G}}\mathcal{B}el_1(\upsilon_1,\upsilon_2,z) & {}_{\mathcal{G}}\mathcal{B}el_2(\upsilon_1,\upsilon_2,z) & \ldots & {}_{\mathcal{G}}\mathcal{B}el_{\varepsilon-1}(\upsilon_1,\upsilon_2,z) & {}_{\mathcal{G}}\mathcal{B}el_\varepsilon(\upsilon_1,\upsilon_2,z) \\ 1 & \frac{1}{2} & \frac{1}{2} & \ldots & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \binom{2}{1}\frac{1}{2} & \ldots & \binom{\varepsilon-1}{1}\frac{1}{2} & \binom{\varepsilon}{1}\frac{1}{2} \\ 0 & 0 & 1 & \ldots & \binom{\varepsilon-1}{2}\frac{1}{2} & \binom{\varepsilon}{2}\frac{1}{2} \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ 0 & 0 & 0 & \ldots & 1 & \binom{\varepsilon}{\varepsilon-1}\frac{1}{2} \end{vmatrix}, \tag{51} $$

$\varepsilon=1,2,3,...,$ where ${}_{\mathcal{G}}\mathcal{B}el_\varepsilon(\upsilon_1,\upsilon_2,z)(\varepsilon=0,1,2,...,)$ are the generalized Bell polynomials of degree ε.

Some other special cases of the generalized Bell-Appell polynomials can be listed below.
III. Taking $\psi(\upsilon_2,\omega)=e^{\upsilon_2\omega^r}$ in generating function (19), gives

$$\mathcal{A}(\omega)\ \exp(\upsilon_1\omega+\upsilon_2\omega^r+z(e^{\omega}-1))=\sum_{\varepsilon=0}^{\infty}{}_{\mathcal{H}^{(r)}\mathcal{B}el}\mathcal{A}_{\varepsilon}(\upsilon_1,\upsilon_2,z)\ \frac{\omega^{\varepsilon}}{\varepsilon!},\tag{52}$$

where ${}_{\mathcal{H}^{(r)}\mathcal{B}el}\mathcal{A}_{\varepsilon}(\upsilon_1,\upsilon_2,z)$ is called the Gould-Hopper-Bell-Appell polynomials.
IV. Taking $\psi(\upsilon_2,\omega)=C_0(\upsilon_2\omega)$ in generating function (19), gives

$$\mathcal{A}(\omega)\ C_0(\upsilon_2\omega)\ \exp(\upsilon_1\omega+z(e^{\omega}-1))=\sum_{\varepsilon=0}^{\infty}{}_{L\mathcal{B}el}\mathcal{A}_{\varepsilon}(\upsilon_1,\upsilon_2,z)\ \frac{\omega^{\varepsilon}}{\varepsilon!},\tag{53}$$

where ${}_{L\mathcal{B}el}\mathcal{A}_{\varepsilon}(\upsilon_1,\upsilon_2,z)$ is called the Laguerre-Bell-Appell polynomials.
V.Taking $\psi(\upsilon_2,\omega)=\frac{1}{1-\upsilon_2\omega^s}$ in generating function (19), gives

$$\frac{\mathcal{A}(\omega)}{1-\upsilon_2\omega^s}\ \exp(\upsilon_1\omega+z(e^{\omega}-1))=\sum_{\varepsilon=0}^{\infty}{}_{e^{(s)}\mathcal{B}el}\mathcal{A}_{\varepsilon}(\upsilon_1,\upsilon_2,z)\ \frac{\omega^{\varepsilon}}{\varepsilon!},\tag{54}$$

where ${}_{e^{(s)}\mathcal{B}el}\mathcal{A}_{\varepsilon}(\upsilon_1,\upsilon_2,z)$ is called the truncated-exponential-Bell-Appell polynomials of order s.
VI. Taking $\psi(\upsilon_2,\omega)=\frac{1}{1-\upsilon_2(e^{\omega}-1)}$ in generating function (19), gives

$$\frac{\mathcal{A}(\omega)}{1-\upsilon_2(e^{\omega}-1)}\ \exp(\upsilon_1\omega+z(e^{\omega}-1))=\sum_{\varepsilon=0}^{\infty}{}_{\mathcal{F}\mathcal{B}el}\mathcal{A}_{\varepsilon}(\upsilon_1,\upsilon_2,z)\ \frac{\omega^{\varepsilon}}{\varepsilon!},\tag{55}$$

where${}_{\mathcal{F}\mathcal{B}el}\mathcal{A}_{\varepsilon}(\upsilon_1,\upsilon_2,z)$ is called the Fubini-Bell-Appell polynomials.
Similarly, by taking different values $\mathcal{A}(\omega)$ in (52)-(55), we can obtain more other special cases of the generalized Bell-Appell polynomials ${}_{\mathcal{G}\mathcal{B}el}\mathcal{A}_{\varepsilon}(\upsilon_1,\upsilon_2,z)$.

5. Conclusions

The hybrid form of special polynomials and numbers has gained worthy considerations by numerous researchers. In this work, we introduced a new class of hybrid special polynomials, namely, the generalized Bell-Appell polynomials. The generating function for the generalized Bell-Appell polynomials and certain related identities and properties are also investigated. The determinant representation is also derived. Further, certain special cases of the generalized Bell-Appell family are also considered. The differential and integral representations containing these types of special polynomials and related applications can be investigated in further studies.

Declarations

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