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Research Paper

On the Scaled Hypercomplex Numbers: Quaternions through Split Quaternions

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Abstract

In this paper, we analyze a certain algebraic structure $\mathcal{H}[-1,1]$ containing all t-scaled hypercomplex numbers of $\mathbb{H}_t$ where the scales t are from -1 to 1, i.e., $-1 \leq t \leq 1$ in $\mathbb{R}$. The algebraic, operator-theoretic, and operator-algebraic properties of $\mathcal{H}[-1,1]$ are studied under the local dynamics on the closed interval $[-1,1]$ inherited from the dynamics on the continuum $\mathbb{R}$. Also, some analytic properties of an interesting type of operators switching scales of hypercomplex numbers acting on $\mathcal{H}[-1,1]$ are considered, and we investigate how they affect the analysis on $\mathcal{H}[-1,1]$.

Key Words: Scaled Hypercomplex Rings, Scaled Hypercomplex Monoids, Dynamical Systems, Free Probability

AMS 2020 Classification: 20G20, 46S10, 47S10

1. Introduction

Hypercomplex numbers are understood to be the pairs $(a,b) \in \mathbb{C}^2$ the complex field $\mathbb{C}$, contained in a ring,

$$\mathbb{H}_t = \left(\mathbb{C}^2, +, \cdot_t\right),$$

for an arbitrarily fixed scale t in the real field $\mathbb{R}$, where $(+)$ is the usual vector-addition on $\mathbb{C}^2$, and $(\cdot_t)$ is the t-scaled vector-multiplication on $\mathbb{C}^2$,

$$(a_1,b_1)\cdot_t(a_2,b_2) = \left(a_1a_2 + tb_1\overline{b_2},\ a_1b_2 + b_1\overline{a_2}\right),$$

where $\overline{w}$ are the conjugates of w in $\mathbb{C}$. By a representation $\left(\mathbb{C}^2, \pi_t\right)$ of the ring $\mathbb{H}_t$, one can understand each hypercomplex number $h=(a,b) \in \mathbb{H}_t$ as a (2×2)-matrix,

$$\pi_t(h) \overset{\text{def}}{=} \begin{pmatrix} a & tb \\ \overline{b} & \overline{a} \end{pmatrix} \text{ in } M_2(\mathbb{C}),$$

canonically, where $M_2(\mathbb{C})$ is the (2×2)-matrix algebra acting on $\mathbb{C}^2$, for all $t \in \mathbb{R}$ (e.g., see [1]). Under our construction, the ring $\mathbb{H}_{-1}$ is the noncommutative field $\mathbb{H}$ of all quaternions (e.g., [2] and [3]), and the ring

$\mathbb{H}_1$ is the ring of all split-quaternion numbers (e.g., [4], [5]). The algebraic, analytic and operator-theoretic properties on $\mathbb{H}_t$, and some free-probabilistic models of $\mathbb{H}_t$ are studied in [1].

In this paper, the family $\{\mathbb{H}_t\}_{t\in\mathbb{R}}$ is considered in a single algebraic structure $\mathscr{H} = \underset{t\in\mathbb{R}}{\oplus^a}\mathbb{H}_t$ dictated by the dynamics of the time-flow $\mathbb{R} = (\mathbb{R}, +)$, where $\oplus^a$ is the pure-algebraic direct product of algebras over the real field $\mathbb{R}$ (in short, $\mathbb{R}$-algebras). We in particular restrict our interests to the sub-family,

$$\{\mathbb{H}_t : -1 \leq t \leq 1\},$$

and the subalgebra,

$$\mathscr{H}[-1,1] = \underset{t\in[-1,1]}{\oplus^a}\mathbb{H}_t \quad \text{of} \quad \mathscr{H},$$

where $[-1,1] = \{s \in \mathbb{R} : -1 \leq s \leq 1\}$ is the closed interval in $\mathbb{R}$. Depending on $-1 \leq t \leq 1$, this operator-algebraic structure $\mathscr{H}[-1,1]$ is regarded as a system starting from the quaternions $\mathbb{H}_{-1}$, ending at the split-quaternions $\mathbb{H}_1$, or vice versa. The reason why we restrict $\mathbb{R}$ to the closed interval $[-1,1]$ (or, restrict $\mathscr{H}$ to $\mathscr{H}[-1,1]$) is because of certain asymptotic analytic data on $\mathbb{H}_t$, especially, where $t \to \infty$, and $t \to -\infty$.

The quaternions $\mathbb{H} = \mathbb{H}_{-1}$ has been studied not only in pure-mathematical areas (e.g., [3], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15]), but also in applied mathematics (e.g., [16], [17], [18], [19] and [20]). Independently, the spectral analysis on $\mathbb{H}$ are considered in [1] and [2], under representation, different from the usual quaternion-eigenvalue problems of quaternion-matrices studied in [11], [17] and [18].

In this paper, certain asymptotic analytic data on $\underset{t\in\mathbb{R}}{\cup}\mathbb{H}_t$ are studied where either $t \to \infty$, or $t \to -\infty$ in $\mathbb{R}$. Those asymptotic data demonstrate that, if $t \to \pm\infty$, then analysis on the $\mathbb{R}$-algebra $\mathbb{H}_t$ seems vague, or undetermined. These motivate us to consider $\mathscr{H}[-1,1]$, from the quaternions $\mathbb{H}_{-1}$ to the split-quaternions $\mathbb{H}_1$. Certain analytic-data-preserving conditions on $\mathscr{H}[-1,1]$ are characterized.

2. The Scaled Hypercomplex Systems $\{\mathbb{H}_t\}_{t\in\mathbb{R}}$

In this section, we review some main results of [1] for our works.

2.1. Scaled Hypercomplex Rings $\{\mathbb{H}_t\}_{t\in\mathbb{R}}$

For $t \in \mathbb{R}$, define the t-scaled vector-multiplication $(\cdot_t)$ on $\mathbb{C}^2$ by

$$(a_1, b_1) \cdot_t (a_2, b_2) \overset{\text{def}}{=} \left(a_1a_2 + tb_1\overline{b_2},\ a_1b_2 + b_1\overline{a_2}\right), \tag{1}$$

for $(a_1, b_1), (a_2, b_2) \in \mathbb{C}^2$. Then the triple $\left(\mathbb{C}^2, +, \cdot_t\right)$ forms a unital ring with its unity $(1, 0)$, where $(+)$ is the vector-addition on $\mathbb{C}^2$, and $(\cdot_t)$ is in the sense of (1). See [1] for details.

Definition 1. For $t \in \mathbb{R}$, we call $\mathbb{H}_t \overset{\text{denote}}{=} \left(\mathbb{C}^2, +, \cdot_t\right)$, the t-scaled hypercomplex ring.

For any $t \in \mathbb{R}$, define an injection,

$$\pi_t : \mathbb{H}_t \to M_2(\mathbb{C}), \tag{2}$$

by

$$\pi_t((a,b)) = \begin{pmatrix} a & tb \\ \overline{b} & \overline{a} \end{pmatrix}, \ \forall (a,b) \in \mathbb{H}_t,$$

where $M_k(\mathbb{C})$ is the matrix algebra of all $(k \times k)$-matrices over $\mathbb{C}$ for all $k \in \mathbb{N}$. This map π_t of (2) satisfies that

$$\pi_t(h_1 + h_2) = \pi_t(h_1) + \pi_t(h_2), \tag{3}$$

and

$$\pi_t(h_1 \cdot_t h_2) = \pi_t(h_1)\pi_t(h_2), \tag{4}$$

in $M_2(\mathbb{C})$, where $\pi_t(h_1)\pi_t(h_2)$ is the matrix multiplication of $\pi_t(h_1)$ and $\pi_t(h_2)$ in $M_2(\mathbb{C})$ (e.g., see [1] for details). By (3)-(4), the pair $\left(\mathbb{C}^2, \pi_t\right)$ forms a representation of $\mathbb{H}_t$. Thus, the realization,

$$\pi_t(\mathbb{H}_t) = \left\{ \begin{pmatrix} a & tb \\ \overline{b} & \overline{a} \end{pmatrix} \in M_2(\mathbb{C}) : (a, b) \in \mathbb{H}_t \right\}, \tag{5}$$

of $\mathbb{H}_t$ is well-defined in $M_2(\mathbb{C})$.

Definition 2. The realization $\mathcal{H}_2^t \overset{\text{denote}}{=} \pi_t(\mathbb{H}_t)$ of (5) is called the t-scaled realization of $\mathbb{H}_t$ (in $M_2(\mathbb{C})$), for a scale $t \in \mathbb{R}$. We denote each element $\pi_t(h)$ by $[h]_t$ in $\mathcal{H}_2^t$, for each $h \in \mathbb{H}_t$.

Remark that the subset,

$$\mathbb{H}_t^{\times} \overset{\text{denote}}{=} \mathbb{H}_t \setminus \{(0,0)\},$$

where $(0,0) \in \mathbb{H}_t$ is the $(+)$-identity of $\mathbb{H}_t$, forms the maximal monoid,

$$\mathbb{H}_t^{\times} \overset{\text{denote}}{=} \left(\mathbb{H}_t^{\times}, \cdot_t\right),$$

with its identity $(1,0)$, in $\mathbb{H}_t$. We call $\mathbb{H}_t^{\times}$, the t-scaled hypercomplex monoid.

2.2. On the t-Scaled Realization $\mathcal{H}_2^t$ of $\mathbb{H}_t$

For any $(a,b) \in \mathbb{H}_t$ realized to be $[(a,b)]_t \in \mathcal{H}_2^t$,

$$\det\left([(a,b)]_t\right) = \det \begin{pmatrix} a & tb \\ \overline{b} & \overline{a} \end{pmatrix} = |a|^2 - t\,|b|^2.$$

where $\det : M_2(\mathbb{C}) \to \mathbb{C}$ is the determinant, and $|.|$ is the modulus on $\mathbb{C}$. Then $|a|^2 \neq t\,|b|^2$ in $\mathbb{C}$, if and only if $[(a,b)]_t$ is invertible "in $\mathcal{H}_2^t$." In particular,

$$(a,b)^{-1} = \left(\frac{\overline{a}}{|a|^2 - t\,|b|^2}, \frac{-b}{|a|^2 - t\,|b|^2} \right) \text{ in } \mathbb{H}_t, \tag{6}$$

satisfying

$$\left[(a,b)^{-1}\right]_t = [(a,b)]_t^{-1} \text{ in } \mathcal{H}_2^t.$$

See [1] for details. The invertibility (6) is meaningful not only in $M_2(\mathbb{C})$, but also "in $\mathcal{H}_2^t$," and hence, "in $\mathbb{H}_t$," as in (6).

Recall that an algebraic structure $(X, +, \cdot)$ is said to be a noncommutative field (or, a skew field), if it is a unital ring, and $(X \setminus \{0_X\}, \cdot)$ forms a non-abelian group, where 0_X is the $(+)$-identity of $(X, +, \cdot)$. (e.g., [1] and [2]).

Proposition 1. *If $t < 0$ in $\mathbb{R}$, then every element (a,b) of the t-scaled hypercomplex monoid $\mathbb{H}_t^{\times}$ are invertible in $\mathbb{H}_t$. The converse also holds, too. i.e.,*

$$t < 0 \text{ in } \mathbb{R} \Longleftrightarrow \mathbb{H}_t \textit{ is a noncommutative field.} \tag{7}$$

Proof. See [1, 2] for details. □

More general to (7), for any scale $t \in \mathbb{R}$, the t-scaled hypercomplex ring $\mathbb{H}_t$ is partitioned by

$$\mathbb{H}_t = \mathbb{H}_t^{inv} \sqcup \mathbb{H}_t^{sing} \tag{8}$$

with

$$\mathbb{H}_t^{inv} = \left\{(a, b) : |a|^2 \neq t\,|b|^2\right\},$$

and

$$\mathbb{H}_t^{sing} = \left\{(a, b) : |a|^2 = t\,|b|^2\right\},$$

where $\sqcup$ is the disjoint union, and hence, the t-scaled hypercomplex monoid $\mathbb{H}_t^{\times}$ is partitioned by

$$\mathbb{H}_t^{\times} = \mathbb{H}_t^{inv} \sqcup \mathbb{H}_t^{\times sing}, \tag{9}$$

with

$$\mathbb{H}_t^{\times sing} = \mathbb{H}_t^{sing} \setminus \{(0, 0)\},$$

by (9). By (7) and (8), the block $\mathbb{H}_t^{inv}$ of (9) is a non-abelian group embedded in $\mathbb{H}_t^{\times}$. Meanwhile, the other block $\mathbb{H}_t^{\times sing}$ of (9) is a semigroup in $\mathbb{H}_t^{\times}$ without identity in $\mathbb{H}_t^{\times}$ (e.g., see [1, 2]).

Definition 3. Let $\mathbb{H}_t^{\times}$ be the t-scaled hypercomplex monoid with its partition (9). The block $\mathbb{H}_t^{inv}$ is called the group-part of $\mathbb{H}_t^{\times}$ (or, of $\mathbb{H}_t$), and the other block $\mathbb{H}_t^{\times sing}$ is called the semigroup-part of $\mathbb{H}_t^{\times}$ (or, of $\mathbb{H}_t$).

By (7), if $t < 0$, then $\mathbb{H}_t = \mathbb{H}_t^{\times} \cup \{(0, 0)\}$, i.e.,

$$t < 0 \Longrightarrow \left[\mathbb{H}_t^{\times sing} \text{ is empty in } \mathbb{H}_t^{\times} \Longleftrightarrow \mathbb{H}_t = \mathbb{H}_t^{inv} \cup \{(0, 0)\}\right], \tag{10}$$

meanwhile, if $t \geq 0$, then $\mathbb{H}_t^{\times sing}$ is a non-empty properly semigroup of $\mathbb{H}_t^{\times}$.

2.3. Spectra of t-Scaled Hypercomplex Numbers

We now review the spectral analysis on $\mathbb{H}_t$ investigated in [1]. Let $(a, b) \in \mathbb{H}_t$ with its realization,

$$\pi_t(a, b) = [(a, b)]_t = \begin{pmatrix} a & tb \\ \overline{b} & \overline{a} \end{pmatrix} \in \mathcal{H}_2^t.$$

Then, in a variable z on $\mathbb{C}$,

$$\det\left([(a, b)]_t - z\,[(1, 0)]_t\right) = z^2 - 2\mathrm{Re}\,(a)\,z + \det\left([a, b]_t\right), \tag{11}$$

where $\mathrm{Re}\,(a)$ is the real part of a in $\mathbb{C}$. This polynomial (11) has its zeroes,

$$z = \mathrm{Re}\,(a) \pm \sqrt{\mathrm{Re}\,(a)^2 - det\left([(a, b)]_t\right)} \tag{12}$$

(e.g., see [1] for details).

Proposition 2. *If $(a, b) \in \mathbb{H}_t$, then the spectrum $spec\left([(a, b)]_t\right)$ of $[(a, h)]_t$ is*

$$spec\left([(a, b)]_t\right) = \left\{\mathrm{Re}\,(a) \pm \sqrt{\mathrm{Re}\,(a)^2 - \det\left([(a, b)]_t\right)}\right\},$$

in $\mathbb{C}$. More precisely, if

$$a = x + yi,\ b = u + vi \in \mathbb{C},$$

with $x, y, u, v \in \mathbb{R}$ and $i = \sqrt{-1}$ in $\mathbb{C}$, then

$$spec\left([(a, b)]_t\right) = \left\{x \pm i\sqrt{y^2 - tu^2 - tv^2}\right\} \text{ in } \mathbb{C}. \tag{13}$$

Proof. The spectrum (13) is obtained by (12). See [1]. □

Observe that if $(a, 0) \in \mathbb{H}_t$, then

$$[(a,0)]_t = \begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix} \text{ in } \mathcal{H}_2^t, \tag{14}$$

satisfying

$$spec\left([(a,0)]_t\right) = \{a, \overline{a}\} \text{ in } \mathbb{C},$$

by (13). Motivated by (13) and (14), define a surjection,

$$\sigma_t : \mathbb{H}_t \to \mathbb{C}, \tag{15}$$

by

$$\sigma_t\left((a,b)\right) \overset{\text{def}}{=} \begin{cases} a = x + yi & \text{if } b = 0 \text{ in } \mathbb{C} \\ x + i\sqrt{y^2 - tu^2 - tv^2} & \text{if } b \neq 0 \text{ in } \mathbb{C}, \end{cases}$$

for all $(a, b) \in \mathbb{H}_t$, with $a = x + yi$ and $b = u + vi$ in $\mathbb{C}$. Note that this surjection σ_t of (15) is not injective.

Definition 4. The surjection $\sigma_t : \mathbb{H}_t \to \mathbb{C}$ of (15) is called the t(-scaled)-spectralization on $\mathbb{H}_t$. The images $\{\sigma_t(\xi)\}_{\xi \in \mathbb{H}_t}$ are said to be t(-scaled)-spectral values.

By the t-spectralization σ_t, one can define the following concept.

Definition 5. Let $\xi \in \mathbb{H}_t$ be a hypercomplex number having its t-spectral value $\sigma_t(\xi) \in \mathbb{C}$. The realization of $(\sigma_t(\xi), 0) \in \mathbb{H}_t$,

$$[(\sigma_t(\xi), 0)]_t = \begin{pmatrix} \sigma_t(\xi) & 0 \\ 0 & \overline{\sigma_t(\xi)} \end{pmatrix} \in \mathcal{H}_2^t,$$

is called the t(-scaled)-spectral form of ξ, denoted by $\Sigma_t(\xi)$ in $\mathcal{H}_2^t$.

Note that the conjugate-notation in the above definition is symbolically understood in the sense that: if

$$\sigma_t\left((a,b)\right) = x + i\sqrt{y^2 - tu^2 - tv^2},$$

with

$$y^2 - tu^2 - tv^2 < 0,$$

where $a = x + yi$ and $b = u + vi$ in $\mathbb{C}$, equivalently, if

$$\sigma_t\left((a,b)\right) = x - \sqrt{tu^2 - tv^2 - y^2} \in \mathbb{R},$$

then the symbol,

$$\overline{\sigma_t\left((a,b)\right)} \overset{\text{means}}{=} \overline{x + i\sqrt{R}} = x - i\sqrt{R} = x + \sqrt{tu^2 - tv^2 - y^2},$$

"in $\mathbb{R}$," where $R = y^2 - tu^2 - tv^2$ in $\mathbb{R}$. Of course, if $R \geq 0$ and hence, if

$$\sigma_t\left((a,b)\right) = x + i\sqrt{R} \text{ in } \mathbb{C},$$

then $\overline{\sigma_t\left((a,b)\right)} = x - i\sqrt{R}$ is the usual conjugate of $\sigma_t\left((a,b)\right)$ in $\mathbb{C}$.

Definition 6. Two hypercomplex numbers $\xi, \eta \in \mathbb{H}_t$ are said to be t(-scaled)-spectral-related, if

$$\sigma_t(\xi) = \sigma_t(\eta) \text{ in } \mathbb{C}.$$

By definition, the t-spectral relation is an equivalence relation on $\mathbb{H}_t$. So, every hypercomplex number ξ of $\mathbb{H}_t$ induces its equivalence class,

$$\widetilde{\xi} \overset{\text{def}}{=} \{\eta \in \mathbb{H}_t : \eta \text{ is } t\text{-spectral related to } \xi\} \text{ in } \mathbb{H}_t,$$

and hence, the quotient set,

$$\widetilde{\mathbb{H}_t} \overset{\text{def}}{=} \left\{ \widetilde{\xi} : \xi \in \mathbb{H}_t \right\}, \tag{16}$$

is well-established. The quotient set $\widetilde{\mathbb{H}_t}$ of (16) is equipotent (or, bijective) to $\mathbb{C}$.

Recall that, in the operator algebra $B(H)$ on a Hilbert space H, two operators T and S are said to be similar in $B(H)$, if there exists an invertible operator $U \in B(H)$, such that

$$S = U^{-1}TU \quad \text{in} \quad B(H).$$

Definition 7. Let $T, S \in \mathcal{H}_2^t$ be realizations of certain hypercomplex numbers of $\mathbb{H}_t$, for $t \in \mathbb{R}$. They are said to be similar "in $\mathcal{H}_2^t$," if there exists an invertible "$U \in \mathcal{H}_2^t$," such that

$$S = U^{-1}TU \text{ in } \mathcal{H}_2^t.$$

Also, hypercomplex numbers ξ and η are said to be similar in $\mathbb{H}_t$, if their realizations $[\xi]_t$ and $[\eta]_t$ are similar in $\mathcal{H}_2^t$.

Let $(a, b) \in \mathbb{H}_t$ with $a = x + yi$ and $b = u + vi$. Then

$$[(a,b)]_t = \begin{pmatrix} a & tb \\ \overline{b} & \overline{a} \end{pmatrix} \in \mathcal{H}_2^t,$$

having its determinant,

$$\det\left([(a,b)]_t\right) = |a|^2 - t\,|b|^2 = \left(x^2 + y^2\right) - t\left(u^2 + v^2\right),$$

meanwhile, the t-spectral form $\Sigma_t((a,b))$ of (a,b) is

$$\Sigma_t((a,b)) = \begin{pmatrix} x + i\sqrt{y^2 - tu^2 - tv^2} & 0 \\ 0 & x - i\sqrt{y^2 - tu^2 - tv^2} \end{pmatrix},$$

in $\mathcal{H}_2^t$, having its determinant,

$$\det\left(\Sigma_t((a,b))\right) = x^2 + \left|y^2 - tu^2 - tv^2\right|.$$

It shows that, $\det\left([(a,b)]_t\right)$ can be negative in $\mathbb{R}$, meanwhile $\det\left(\Sigma_t((a,b))\right)$ is always non-negative, for some $t \in \mathbb{R}$, i.e.,

$$\det\left([(a,b)]_t\right) \neq \det\left(\Sigma_t((a,b))\right), \text{ in general.}$$

It implies that $[(a,b)]_t$ and $\Sigma_t((a,b))$ are not similar in $\mathcal{H}_2^t$, in general, for some $t \in \mathbb{R}$.

Lemma 1. If $t < 0$ in $\mathbb{R}$, then every hypercomplex number $h \in \mathbb{H}_t$ is similar to $(\sigma_t(h), 0) \in \mathbb{H}_t$, where $\sigma_t(h)$ is the t-spectral value of h. Equivalently,

$$t < 0 \Longrightarrow [h]_t \text{ and } \Sigma_t(h) \tag{17}$$

are similar in $\mathcal{H}_2^t$.

Proof. If $h = (a, 0) \in \mathbb{H}_t$, where $t < 0$, then

$$[(a,0)]_t = \begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix} = \Sigma_t\left((a,0)\right) \text{ in } \mathcal{H}_2^t,$$

since $\sigma_t\left((a,0)\right) = a$ in $\mathbb{C}$. Therefore, $[(a,0)]_t$ and $\Sigma_t\left((a,0)\right)$ are clearly similar in $\mathcal{H}_2^t$. Meanwhile, if $h = (a, b) \in \mathbb{H}_t$ with $b \neq 0$, then $[h]_t$ and $\Sigma_t\left(h\right)$ are similar in $\mathcal{H}_2^t$, because there exists

$$q_h = \left(1, \ \overline{\frac{w-a}{tb}}\right) \in \mathbb{H}_t,$$

such that

$$\Sigma_t\left(h\right) = [q_h]_t^{-1}\,[h]_t\,[q_h]_t \ \text{ in } \ \mathcal{H}_2^t,$$

for any $w \in \mathbb{C} \setminus \{0\}$ (e.g., see [1] for details). Therefore, if $t < 0$, then $[h]_t$ and $\Sigma_t\left(h\right)$ are similar in $\mathcal{H}_2^t$, for "all" $h \in \mathbb{H}_t$. □

By (17), we obtain the following result in [1, 2].

Proposition 3. *Suppose $t < 0$ in $\mathbb{R}$. Then*

$$t < 0 \Longrightarrow [t\text{-spectral relation} \overset{equi}{=} \text{similarity on } \mathbb{H}_t,]$$

where "$\overset{equi}{=}$" means "being equivalent to, as equivalence relations."

Proof. See [1, 2] in details. □

2.4. Scaled Hypercomplex Rings $\mathbb{H}_t$ as $\mathbb{R}$-Vector Spaces

From below, for convenience, we denote the t-scaled multiplication $(\cdot_t)$ simply by $(\cdot)$ if there are no confusions, i.e.,

$$h_1 h_2 \overset{\text{denote}}{=} h_1 \cdot_t h_2 \ \text{ in } \mathbb{H}_t, \ \ \forall h_1, h_2 \in \mathbb{H}_t.$$

In this section, we define a vector space,

$$\mathscr{H}_t = \text{span}_{\mathbb{R}}\left(\{1, i, j_t, k_t\}\right), \tag{18}$$

generated by its basis,

$$\mathcal{B}_t = \{1, i, j_t, k_t\},$$

over the real field $\mathbb{R}$, under the relation on $\mathcal{B}_t$:

$$i^2 = -1, \ j_t^2 = t = k_t^2, \tag{19}$$

$$\begin{array}{ccc} & i & \\ {}^{1}\swarrow & & \nwarrow^{-t} \\ j_t & \underset{1}{\longrightarrow} & k_t \end{array} \quad \text{and} \quad \begin{array}{ccc} & i & \\ {}^{t}\nearrow & & \searrow^{-1} \\ j_t & \underset{-1}{\longleftarrow} & k_t \end{array},$$

where the first diagram means that

$$i j_t = k_t, \ \ j_t k_t = -ti, \ \ k_t i = j_t,$$

and the second diagram means that

$$j_t i = -k_t, \ \ k_t j_t = ti, \ \ i k_t = -j_t.$$

i.e., the set $\mathscr{H}_t$ of (18) is a vector space over $\mathbb{R}$ (in short, a $\mathbb{R}$-vector space) whose $\mathbb{R}$-basis $\mathcal{B}_t$ satisfies the relation (19).

Lemma 2. Let $\mathbb{H}_t$ be the t-scaled hypercomplex ring for $t \in \mathbb{R}$. Then it is a $\mathbb{R}$-vector space,

$$\mathbb{H}_t = \text{span}_{\mathbb{R}} \{\mathbf{1}, \mathbf{i}, \mathbf{j}_t, \mathbf{k}_t\}, \tag{20}$$

$$\mathbf{1} = (1,0), \ \mathbf{i} = (i,0), \ \mathbf{j}_t = (0,1), \text{ and } \mathbf{k}_t = (0,i).$$

And the basis elements $\mathbf{i}$, $\mathbf{j}_t$ and $\mathbf{k}_t$ of (20) satisfies that

$$\begin{aligned} \mathbf{i}^2 &= -\mathbf{1}, \ \mathbf{j}_t^2 = t\mathbf{1} = \mathbf{k}_t^2, \\ \mathbf{i}\mathbf{j}_t &= \mathbf{k}_t, \ \mathbf{k}_t\mathbf{j}_t = -t\mathbf{i}, \ \mathbf{k}_t\mathbf{i} = \mathbf{j}_t, \\ \mathbf{i}\mathbf{k}_t &= -\mathbf{j}_t, \ \mathbf{j}_t\mathbf{k}_t = t\mathbf{i}, \ \mathbf{j}_t\mathbf{i} = -\mathbf{k}_t. \end{aligned} \tag{21}$$

Proof. See [2] for details. □

By (20) and (21), we have the following result.

Proposition 4. *Every t-scaled hypercomplex ring $\mathbb{H}_t$ is isomorphic to the $\mathbb{R}$-vector space $\mathscr{H}_t = \text{span}_{\mathbb{R}}\mathcal{B}_t$ of (18) whose $\mathbb{R}$-basis $\mathcal{B}_t$ satisfies the relation (19), for all $t \in \mathbb{R}$.*

Proof. As a $\mathbb{R}$-vector space (20), the t-scaled hypercomplex ring $\mathbb{H}_t$ satisfies

$$\mathbb{H}_t = \text{span}_{\mathbb{R}} \{\mathbf{1}, \mathbf{i}, \mathbf{j}_t, \mathbf{k}_t\}.$$

Then one can define the $\mathbb{R}$-basis-preserving bijection $\Phi : \mathbb{H}_t \to \mathscr{H}_t$ by

$$\Phi(x\mathbf{1} + y\mathbf{i} + u\mathbf{j}_t + v\mathbf{k}_t) = x + yi + uj_t + vk_t,$$

in $\mathscr{H}_t$. □

By the above structure theorem, one can re-define $\mathbb{H}_t$ as follows.

Definition 8. Re-define our t-scaled hypercomplex ring $\mathbb{H}_t$ by

$$\mathbb{H}_t \overset{\text{def}}{=} \left\{ x + yi + uj_t + vk_t \;\middle|\; \begin{array}{c} x, y, u, v \in \mathbb{R} \\ i^2 = -1, \quad j_t^2 = t = k_t^2 \\ ij_t = k_t, \ j_tk_t = -ti, \ k_ti = j_t \\ ik_t = -j_t, \ k_tj_t = ti, \ j_ti = -k_t \end{array} \right\}, \tag{22}$$

as a $\mathbb{R}$-vector space $\text{span}_{\mathbb{R}} \{1, i, j_t, k_t\}$.

Remark 1. Note that if we understand $\mathbb{H}_t$ as the t-scaled hypercomplex ring, then each element h of $\mathbb{H}_t$ is regarded as a (2×2)-matrix $[h]_t \in \mathcal{H}_2^t$, under its Hilbert-space representation $(\mathbb{C}^2, \pi_t)$ "over $\mathbb{C}$." Meanwhile, if we regard $\mathbb{H}_t$ as the vector space (22), then every element h of $\mathbb{H}_t$ is a vector in $\text{span}_{\mathbb{R}} \{1, i, j_t, k_t\}$, "over $\mathbb{R}$." Note that

$$\mathbb{H}_t \ni x + yi + uj_t + vk_t = (x + yi) + (u + vi) j_t,$$

since $ij_t = k_t$ by (22). i.e., $h = (a, b) \in \mathbb{H}_t$ with $a, b \in \mathbb{C}$, if and only if $h = a + bj_t \in \mathbb{H}_t$. So, we will use the notations,

$$(a, b), \quad \text{or} \quad a + bj_t, \quad \text{in} \quad \mathbb{H}_t,$$

alternatively from below.

Proposition 5. *For any scale $t \in \mathbb{R}$, the t-scaled hypercomplexes,*

$$\mathbb{H}_t \textit{ is an algebra over } \mathbb{R} \textit{ (in short, a } \mathbb{R}\textit{-algebra).} \tag{23}$$

Proof. Since $\mathbb{H}_t$ is both a ring and a $\mathbb{R}$-vector space, it forms a $\mathbb{R}$-algebra. □

Remark and Notation 1. As we have seen in Section 2, the set $\mathbb{H}_t$ of all t-scaled hypercomplex numbers is a unital ring algebraically; and it is a $\mathbb{R}$-vector space analytically; and it forms a $\mathbb{R}$-algebra operator-algebraically. So, from below, we call $\mathbb{H}_t$, the t-scaled hypercomplexes as a ring, or a $\mathbb{R}$-vector space, or a $\mathbb{R}$-algebra, case-by-case.

Recall that, in [2], we restricted the normalized trace $\tau = \frac{1}{2}tr$ on $M_2(\mathbb{C})$, where tr is the usual trace on $M_2(\mathbb{C})$, to the t-scaled realization $\mathcal{H}_2^t$, i.e.,

$$\tau\left([(a, b)]_t\right) = \tau\begin{pmatrix} a & tb \\ \overline{b} & \overline{a} \end{pmatrix} = \frac{a + \overline{a}}{2} = \mathrm{Re}\,(a)\,,$$

implying the existence of a trace, also denoted by τ_t, on $\mathbb{H}_t$,

$$\tau_t\left((a, b)\right) = \mathrm{Re}\,(a)\,, \quad \forall\,(a, b) \in \mathbb{H}_t.$$

Also, see Section 4 below. Note here that even though τ is a trace on $M_2(\mathbb{C})$ over $\mathbb{C}$, the restriction τ is a linear functional on $\mathbb{H}_t$ "over $\mathbb{R}$." From this trace τ on $\mathbb{H}_t$, we defined a definite, or indefinite semi-inner product $\langle , \rangle_t$ on $\mathbb{H}_t$ over $\mathbb{R}$, by

$$\langle h_1, h_2\rangle_t \overset{\text{def}}{=} \tau\left(h_1 h_2^{\dagger}\right), \quad \forall h_1, h_2 \in \mathbb{H}_t.$$

In particular, it forms a definite inner product if $t < 0$, or an indefinite inner product if $t > 0$, or an indefinite semi-inner product if $t = 0$ (See [2]). So, one can get the semi-norm $\|.\|_t$,

$$\|h\|_t \overset{\text{def}}{=} \sqrt{\left|\langle h, h\rangle_t\right|}, \quad \forall h \in \mathbb{H}_t,$$

where $|.|$ is the absolute value on $\mathbb{R}$, making $\mathbb{H}_t$ as a $\mathbb{R}$-Hilbert space if $t < 0$, or a complete $\mathbb{R}$-semi-normed space if $t \geq 0$ (See [2]).

However, in this paper, we simply understand the t-scaled hypercomplexes $\mathbb{H}_t$ as a $\mathbb{R}$-algebra equipped with the usual 4-dimensional $\mathbb{R}$-vector space norm $\|.\|_4$. i.e., we define a norm $\|.\|_4$ on the $\mathbb{R}$-vector space $\mathbb{H}_t = \mathrm{span}_{\mathbb{R}}\{1, i, j_t, k_t\}$ simply by

$$\|x + yi + uj_t + vk_t\|_4 \overset{\text{def}}{=} \|(x, y, u, v)\|_4 = \sqrt{x^2 + y^2 + u^2 + v^2}, \tag{24}$$

where $\|.\|_4$ in the first equality is the usual norm on $\mathbb{R}^4$. Then one can understand $\mathbb{H}_t$ as a Banach space, for all $t \in \mathbb{R}$. Recall and note that in a finite-dimensional vector space (over $\mathbb{R}$, or over $\mathbb{C}$), all norms are equivalent (e.g., [21, 22]), and hence, the above norm $\|.\|_4$ is well-defined on the 4-dimensional $\mathbb{R}$-vector space $\mathbb{H}_t$. So, $\mathbb{H}_t$ forms a Banach algebra over $\mathbb{R}$ (in short, a $\mathbb{R}$-Banach algebra).

Corollary 1. If $\|.\|_4$ is the norm (24) on the t-scaled hypercomplexes $\mathbb{H}_t$, then $\mathbb{H}_t$ forms a $\mathbb{R}$-Banach algebra, for all $t \in \mathbb{R}$.

Proof. By (23), the t-scaled hypercomplexes $\mathbb{H}_t$ forms a $\mathbb{R}$-algebra. As we discussed in the above paragraph, the norm $\|.\|_4$ of (24) is well-defined on the 4-dimensional $\mathbb{R}$-vector space $\mathbb{H}_t = \mathrm{span}_{\mathbb{R}}\{1, i, j_t, k_t\}$. Since every norm on a finite-dimensional vector space is complete (e.g., [21, 22]), this norm $\|.\|_4$ is complete on $\mathbb{H}_t$, i.e., $\mathbb{H}_t$ forms a $\mathbb{R}$-Banach space. So, as an algebra, $\mathbb{H}_t$ is a $\mathbb{R}$-Banach algebra. □

3. Scale-Shift Operators $\{S_{t,s} : \mathbb{H}_t \to \mathbb{H}_s\}_{t,s\in\mathbb{R}}$

In this section, we consider the t-scaled hypercomplexes $\mathbb{H}_t$ as a $\mathbb{R}$-Banach algebra equipped with the norm $\|.\|_4$ of (24), for all $t \in \mathbb{R}$. Define functions,

$$S_{t_1,t_2} : \mathbb{H}_{t_1} \to \mathbb{H}_{t_2}, \tag{25}$$

by

$$S_{t_1,t_2}\left(x + yi + uj_{t_1} + vk_{t_1}\right) \overset{\text{def}}{=} x + yi + uj_{t_2} + vk_{t_2},$$

in $\mathbb{H}_{t_2}$, for all $x+yi+uj_{t_1}+vk_{t_1} \in \mathbb{H}_{t_1}$ with $x, y, u, v \in \mathbb{R}$, for any $t_1, t_2 \in \mathbb{R}$. Indeed, the function S_{t_1,t_2} of (25) is a well-defined bijective function from $\mathbb{H}_{t_1}$ onto $\mathbb{H}_{t_2}$, because it is $\mathbb{R}$-basis-preserving map. Moreover, it is a $\mathbb{R}$-linear transformation because

$$S_{t_1,t_2}\left(r_1h_1 + r_2h_2\right) = r_1S_{t_1,t_2}\left(h_1\right) + r_2S_{t_1,t_2}\left(h_2\right),$$

in $\mathbb{H}_{t_2}$, for all $r_1, r_2 \in \mathbb{R}$ and $h_1, h_2 \in \mathbb{H}_{t_1}$, for $t_1, t_2 \in \mathbb{R}$. By the definition (25), if $t_1 = t = t_2$ in $\mathbb{R}$, then $S_{t,t}$ is the identity $\mathbb{R}$-linear transformation I_t, i.e., $I_t(h) = h = S_{t,t}(h)$, for all $h \in \mathbb{H}_t$. Note that, by the finite-dimensionality of $\{\mathbb{H}_t\}_{t\in\mathbb{R}}$ of (22) over $\mathbb{R}$, this bijective $\mathbb{R}$-linear transformations $\{S_{t_1,t_2}\}_{t_1,t_2\in\mathbb{R}}$ are bounded (or, continuous under $\mathbb{R}$-linearity).

Lemma 3. The functions $\{S_{t_1,t_2} : \mathbb{H}_{t_1} \to \mathbb{H}_{t_2}\}_{t_1,t_2\in\mathbb{R}}$ of (25) are $\mathbb{R}$-Banach-space-isomorphisms.

Proof. It is shown by the very definition (25), since the bijection S_{t_1,t_2} preserves the basis $\{1, i, j_{t_1}, k_{t_1}\}$ of $\mathbb{H}_{t_1}$ onto the basis $\{1, i, j_{t_2}, k_{t_2}\}$ of $\mathbb{H}_{t_2}$ under $\mathbb{R}$-linearity, for any $t_1, t_2 \in \mathbb{R}$. The boundedness is guaranteed since

$$\|x + yi + uj_{t_2} + vk_{t_2}\|_4 = \|(x, y, u, v)\|_4 = \|x + yi + uj_{t_1} + vk_{t_1}\|_4,$$

for all $x, y, u, v \in \mathbb{R}$. ☐

If we consider $\mathbb{H}_t$ as a unital ring $\left(\mathbb{C}^2, +, \cdot\right) \overset{\text{denote}}{=} \left(\mathbb{C}^2, +, \cdot_t\right)$ for any $t \in \mathbb{R}$, then the $\mathbb{R}$-isomorphism $\{S_{t_1,t_2}\}_{t_1,t_2\in\mathbb{R}}$ of (25) is understood to be the morphisms,

$$S_{t_1,t_2}\left((a, b)\right) = (a, b) \in \mathbb{H}_{t_2},\ \forall\,(a, b) \in \mathbb{H}_{t_1} \tag{26}$$

for all $t_1, t_2 \in \mathbb{R}$, as topological-ring-isomorphisms (or, continuous ring-isomorphisms), inducing the equivalent topological-ring-isomorphisms, also denoted by

$$\left\{S_{t_1,t_2} : \mathcal{H}_2^{t_1} \to \mathcal{H}_2^{t_2}\right\}_{t_1,t_2\in\mathbb{R}}, \tag{27}$$

satisfying

$$S_{t_1,t_2}\begin{pmatrix} a & t_1b \\ \overline{b} & \overline{a} \end{pmatrix} = \begin{pmatrix} a & t_2b \\ \overline{b} & \overline{a} \end{pmatrix} \in \mathcal{H}_2^{t_1},\ \forall \begin{pmatrix} a & t_1b \\ \overline{b} & \overline{a} \end{pmatrix} \in \mathcal{H}_2^{t_1},$$

for all $t_1, t_2 \in \mathbb{R}$.

Lemma 4. The $\mathbb{R}$-isomorphisms $\{S_{t_1,t_2}\}_{t_1,t_2}$ are topological-ring-isomorphisms in the sense that:

$$S_{t_1,t_2} : (a, b) \in \mathbb{H}_{t_1} \longmapsto (a, b) \in \mathbb{H}_{t_2}, \tag{28}$$

and

$$S_{t_1,t_2} : \begin{pmatrix} a & t_1b \\ \overline{b} & \overline{a} \end{pmatrix} \in \mathcal{H}_2^{t_1} \longmapsto \begin{pmatrix} a & t_2b \\ \overline{b} & \overline{a} \end{pmatrix} \in \mathcal{H}_2^{t_2},$$

for all $(a, b) \in \mathbb{C}^2$, and $t_1, t_2 \in \mathbb{R}$.

Proof. The functions in (28) are well-defined by (26) and (27), respectively. Since $\mathbb{H}_t$ and $\mathcal{H}_2^t$ are isomorphic rings under the representations $\left(\mathbb{C}^2, \pi_t\right)$ for all $t \in \mathbb{R}$, it is enough to check that S_{t_1,t_2} of (27) is a topological-ring-isomorphism from $\mathbb{H}_{t_1}$ onto $\mathbb{H}_{t_2}$, for all $t_1, t_2 \in \mathbb{R}$. Let's fix $t_1, t_2 \in \mathbb{R}$. Then, since S_{t_1,t_2} of (25) is a $\mathbb{R}$-isomorphism, the function S_{t_1,t_2} of (27) is a bijective. It is indeed a topological-ring-isomorphism satisfying

$$S_{t_1,t_2}\left([h_1]_{t_1}[h_2]_{t_1}\right) = S_{t_1,t_2}\left([h_1]_{t_1}\right) S_{t_1,t_2}\left([h_2]_{t_1}\right),$$

in $\mathcal{H}_2^{t_2}$, for $h_1, h_2 \in \mathbb{H}_{t_1}$. Indeed, one has

$$S_{t_1,t_2}\left([h_1]_{t_1}[h_2]_{t_1}\right) = S_{t_1,t_2}\left([h_1 h_2]_{t_1}\right)$$

where

$$\begin{aligned} h_1 h_2 &\overset{\text{denote}}{=} h_1 \cdot_{t_1} h_2 \text{ in } \mathbb{H}_{t_2} \\ &= [h_1 h_2]_{t_2} \end{aligned}$$

where

$$\begin{aligned} h_1 h_2 &= h_1 \cdot_{t_2} h_2 \text{ in } \mathbb{H}_{t_1} \\ &= [h_1]_{t_2}[h_2]_{t_2} = \left(S_{t_1,t_2}\left([h_1]_{t_1}\right)\right)\left(S_{t_1,t_2}\left([h_2]_{t_1}\right)\right) \end{aligned}$$

in $\mathcal{H}_2^{t_2}$. It shows that this bijection S_{t_1,t_2} of (27) is a ring-homomorphism, and hence, a ring-isomorphism from $\mathcal{H}_2^{t_1}$ onto $\mathcal{H}_2^{t_2}$. The continuity of this ring-isomorphism S_{t_1,t_2} of (27) is guaranteed by that of the $\mathbb{R}$-isomorphism S_{t_1,t_2} of (25). □

By the above two lemmas, one can conclude the following result.

Theorem 6. *The bijections* $\{S_{t_1,t_2} : \mathbb{H}_{t_1} \to \mathbb{H}_{t_2}\}_{t_1,t_2 \in \mathbb{R}}$ *of* (25) *are* $\mathbb{R}$*-Banach-algebra-isomorphisms.*

Proof. Since $\mathbb{H}_t$ is both a topological ring $\left(\mathbb{C}^2, +, \cdot\right)$ and a $\mathbb{R}$-Banach space $\text{span}_{\mathbb{R}}\{1, i, j_t, k_t\}$, it forms a $\mathbb{R}$-Banach algebra, in particular, the completeness of $\mathbb{H}_t$ is guaranteed by its finite-dimensionality of $\mathbb{H}_t$ over $\mathbb{R}$, for all $t \in \mathbb{R}$. By the two lemmas, the bijections $\{S_{t_1,t_2}\}_{t_1,t_2 \in \mathbb{R}}$ are bijective bounded multiplicative $\mathbb{R}$-linear transformations, equivalently, they form bounded $\mathbb{R}$-algebra-isomorphisms. Finally, consider that, for all $h = x + yi + uj_{t_1} + vk_{t_1} \in \mathbb{H}_{t_1}$, we have

$$\|h\|_4 = \sqrt{x^2 + y^2 + u^2 + v^2}, \text{ in } \mathbb{H}_{t_1},$$

and

$$\|S_{t_1,t_2}(h)\|_4 = \|x + yi + uj_{t_2} + vk_{t_2}\|_4 = \sqrt{x^2 + y^2 + u^2 + v^2},$$

in $\mathbb{H}_{t_2}$, implying that the bounded $\mathbb{R}$-algebra-isomorphism S_{t_1,t_2} is isometric from $\mathbb{H}_{t_1}$ onto $\mathbb{H}_{t_2}$, for all $t_1, t_2 \in \mathbb{R}$. □

The above theorem shows that the bijections $\{S_{t_1,t_2}\}_{t_1,t_2 \in \mathbb{R}}$ of (25) preserves operator-algebraic structures of the scaled hypercomplexes $\{\mathbb{H}_t\}_{t \in \mathbb{R}}$ by interchanging scales. Also, one can verify that

$$S_{t_1,t_2}^{-1} = S_{t_2,t_1}, \quad \forall t_1, t_2 \in \mathbb{R}$$

Definition 9. The $\mathbb{R}$-Banach-algebra-isomorphisms S_{t_1,t_2} of (25) are called the scale-shift operators from t_1 to t_2 (in short, (t_1, t_2)-shifts), for all $t_1, t_2 \in \mathbb{R}$.

By the above theorem, all (t_1, t_2)-shifts S_{t_1,t_2} are $\mathbb{R}$-Banach-algebra-isomorphisms from $\mathbb{H}_{t_1}$ onto $\mathbb{H}_{t_2}$ for all $t_1, t_2 \in \mathbb{R}$.

Remark 3.1. As we have seen above, our scale-shifts $\{S_{t_1,t_2}\}_{t_1,t_2 \in \mathbb{R}}$ are $\mathbb{R}$-Banach-algebra-isomorphisms. So, the (t_1, t_2)-shift S_{t_1,t_2} preserves $\mathbb{R}$-Banach-algebraic properties of $\mathbb{H}_{t_1}$ onto those of $\mathbb{H}_{t_2}$, for any $t_1, t_2 \in \mathbb{R}$. Meanwhile, the invertibility and spectral properties on "the t-scaled hypercomplex ring" $\mathbb{H}_t$ are considered "over $\mathbb{C}$," for $t \in \mathbb{R}$. Thus, one cannot confirm that our $\mathbb{R}$-Banach-algebra isomorphisms $\{S_{t_1,t_2}\}_{t_1,t_2 \in \mathbb{R}}$ acting "over $\mathbb{R}$," preserve these invertibility and the spectral properties of Section 2 "over $\mathbb{C}$."

Proposition 7. *Let $h = (a, b) \in \mathbb{H}_t$, and $S_{t,s}$, the (t, s)-shift. Then $S_{t,s}(h)$ is invertible in $\mathbb{H}_s$, if and only if $S_{t,s}(h) \in \mathbb{H}_s^{inv}$, if and only if*

$$|a|^2 - s\,|b|^2 \neq 0.$$

Proof. By definition, $S_{t,s}(h) = (a, b) \in \mathbb{H}_s$. So, it is invertible in $\mathbb{H}_s$, if and only if it is contained in the group-part $\mathbb{H}_s^{inv}$ of $\mathbb{H}_s$, if and only if

$$det\left([(a,b)]_s\right) = |a|^2 - s\,|b|^2 \neq 0.$$

□

The above proposition illustrates that the invertibility on $\mathbb{H}_t$ is not preserved by the (t, s)-shift $S_{t,s}$, in general. For instance, suppose $(a, b) \in \mathbb{H}_t^{\times}$ satisfies

$$|a|^2 - t\,|b|^2 = 0,$$

in $\mathbb{H}_t$, equivalently, $(a, b) \in \mathbb{H}_t^{\times sing}$. It means that $t \geq 0$ by (8). If $s < 0$, then

$$S_{t,s}\left((a,b)\right) = (a,b) \in \mathbb{H}_s^{\times},$$

and hence,

$$\det\left([(a,b)]_s\right) = |a|^2 - s\,|b|^2 > 0,$$

implying that $S_{t,s}((a, b)) \in \mathbb{H}_s^{inv}$ in $\mathbb{H}_s$. So, indeed, our scale-shifts do not preserve the invertibility on scaled hypercomplexes, in general. It shows that our (t, s)-shift $S_{t,s}$ "does" preserve the ring-structures of $\mathbb{H}_t$ to those of $\mathbb{H}_s$, however, it does not preserve noncommutative-field structure of $\mathbb{H}_s$ (where $s < 0$) to that of $\mathbb{H}_t$ (where $t \geq 0$).

Theorem 8. *Suppose $t, s < 0$ in $\mathbb{R}$. Then*

$$h \in \mathbb{H}_t \text{ is invertible} \iff S_{t,s}(h) \in \mathbb{H}_s \text{ is invertible} \tag{29}$$

Proof. Assume that $t, s < 0$ in $\mathbb{R}$, and let $S_{t,s}$ be the (t, s)-shift. Since $t, s < 0$, one has

$$\mathbb{H}_r = \mathbb{H}_r^{inv} \cup \{(0,0)\}, \quad \forall r = t, s,$$

by (10). Thus, if $h \neq (0, 0)$ in $\mathbb{H}_t$, then $S_{t,s}(h) \neq (0, 0)$ in $\mathbb{H}_s$; and if $q \neq (0, 0)$ in $\mathbb{H}_s$, then $S_{t,s}^{-1}(q) = S_{s,t}(q) \neq (0, 0)$ in $\mathbb{H}_t$. Therefore, the invertibility (29) holds. □

Also, we have the following result.

Theorem 9. *Suppose $t \geq 0$ and $s < 0$, and assume that $h = (a, b) \in \mathbb{H}_t^{\times}$ in $\mathbb{H}_t$. Then $S_{t,s}(h)$ is invertible in $\mathbb{H}_s$. As a special case, if $h \in \mathbb{H}_t^{inv}$, then $S_{t,s}(h) \in \mathbb{H}_s^{inv}$ in $\mathbb{H}_s$. i.e.,*

$$t \geq 0,\ s < 0 \Longrightarrow \left[h \text{ is non-zero} \Longrightarrow S_{t,s}(h) \text{ is invertible}\right], \tag{30}$$

and hence,

$$t \geq 0,\ s < 0 \Longrightarrow \left[h \text{ is invertible} \Longrightarrow S_{t,s}(h) \text{ is invertible}\right].$$

Proof. Recall that if $t \geq 0$, then $\mathbb{H}_t = \mathbb{H}_t^{inv} \sqcup \mathbb{H}_t^{sing}$, with non-empty block $\mathbb{H}_t^{sing}$; and if $s < 0$, then $\mathbb{H}_s = \mathbb{H}_s^{inv} \sqcup \{(0,0)\}$ by (8) and (10). Since our (t,s)-shift $S_{t,s}$ is a $\mathbb{R}$-Banach-algebra isomorphism, it assign $(0,0) \in \mathbb{H}_t$ to $(0,0) \in \mathbb{H}_s$. Thus, by (8) and (10),

$$S_{t,s}\left(\mathbb{H}_t^{\times}\right) = \mathbb{H}_s^{inv}.$$

So, the first statement of (30) holds. So, as a special case of this first statement, the second statement of (30) immediately holds true, too. $\square$

The above relation (30) shows that if $t \geq 0$ and $s < 0$, then the invertibility on $\mathbb{H}_t$ is preserved to be that on $\mathbb{H}_s$ via the (t,s)-shift $S_{t,s}$. However, the (t,s)-shift $S_{t,s}$ actually assigns all non-invertible non-zero elements of $\mathbb{H}_t$ to invertible elements of $\mathbb{H}_s$, too, by (30).

Theorem 10. *Let $t, s \geq 0$, and suppose $h = (a,b) \in \mathbb{H}_t^{inv}$ is invertible in $\mathbb{H}_t$. If $|a|^2 > t|b|^2$ and if $s \leq t$, then $S_{t,s}(h) \in \mathbb{H}_s^{inv}$ is invertible in $\mathbb{H}_s$. Similarly, if $|a|^2 < t|b|^2$ and it $s \geq t$, then $S_{t,s}(h) \in \mathbb{H}_s^{inv}$ is invertible in $\mathbb{H}_s$. i.e.,*

$$|a|^2 > t|b|^2, \; s \leq t \Longrightarrow S_{t,s}(a,b) \in \mathbb{H}_s^{inv}, \tag{31}$$

and

$$|a|^2 < t|b|^2, \; s \geq t \Longrightarrow S_{t,s}(a,b) \in \mathbb{H}_s^{inv}.$$

Proof. Assume that $t, s \geq 0$ and $h = (a,b) \in \mathbb{H}_t^{inv}$ in $\mathbb{H}_t$. By its invertibility in $\mathbb{H}_t$,

$$\det\left([h]_t\right) = |a|^2 - t|b|^2 \neq 0,$$

if and only if

$$\text{either } |a|^2 > t|b|^2, \text{ or } |a|^2 < t|b|^2.$$

Also, the corresponding s-scaled hypercomplex number $S_{t,s}(h) \in \mathbb{H}_s$ satisfies that

$$\det\left([S_{t,s}(h)]_s\right) = \det\begin{pmatrix} a & sb \\ \overline{b} & \overline{a} \end{pmatrix} = |a|^2 - s|b|^2.$$

So, if $|a|^2 > t|b|^2$, and $s \leq t$, then

$$|a|^2 > t|b|^2 \geq s|b|^2 \Longrightarrow |a|^2 > s|b|^2 \Longrightarrow |a|^2 - s|b|^2 \neq 0,$$

and hence, $S_{t,s}(h) \in \mathbb{H}_s^{inv}$ is invertible in $\mathbb{H}_s$. Similarly, if $|a|^2 < t|b|^2$ and $s \geq t$, then

$$|a|^2 < t|b|^2 \leq s|b|^2 \Longrightarrow |a|^2 < s|b|^2 \Longrightarrow |a|^2 - s|b|^2 \neq 0,$$

and hence, $S_{t,s}(h) \in \mathbb{H}_s^{inv}$ is invertible in $\mathbb{H}_s$. Therefore, the relations of (31) hold. $\square$

The above three theorems characterizes some cases where our (t,s)-shift $S_{t,s}$ preserves the invertibility on $\mathbb{H}_t$ to that on $\mathbb{H}_s$, even though it does not preserve in general.

Then how about the spectral properties on $\{\mathbb{H}_t\}_{t \in \mathbb{R}}$?

Proposition 11. *Let $h = (a,b) \in \mathbb{H}_t$, with $a = x + yi$ and $b = u + vi$ in $\mathbb{C}$, and $S_{t,s}$, the (t,s)-shift. Then*

$$\sigma_s\left(S_{t,s}(h)\right) = x + i\sqrt{y^2 - su^2 - sv^2}, \tag{32}$$

and hence,

$$\text{spec}\left(S_{t,s}(h)\right) = \left\{x \pm i\sqrt{y^2 - su^2 - sv^2}\right\} \text{ in } \mathbb{C}.$$

Proof. Since $S_{t,s}(h) = (a,b) \in \mathbb{H}_s$, for $h = (a,b) \in \mathbb{H}_t$, its s-spectral value satisfies

$$\sigma_s((a,b)) = x + i\sqrt{y^2 - su^2 - sv^2} \overset{\text{denote}}{=} z,$$

in $\mathbb{C}$, and hence,

$$\text{spec}((a,b)) = \{z, \ \overline{z}\} \text{ in } \mathbb{C},$$

by (13) and (15). □

The proof of (32) illustrates that t-spectral values on $\mathbb{H}_t$ are not preserved by those on $\mathbb{H}_s$, under the action of the (t,s)-shift $S_{t,s}$, whenever $t \neq s$ in $\mathbb{R}$. For example, let's assume that $(a,b) \in \mathbb{H}_t^{\times}$, with $a = x + yi$ and $b = u + vi$ in $\mathbb{C}$, satisfies

$$\sigma_t((a,b)) = x + i\sqrt{y^2 - tu^2 - tv^2} = x - \sqrt{tu^2 + tv^2 - y^2} \in \mathbb{R},$$

equivalently, $y^2 - tu^2 - tv^2 < 0$ for t. If

$$y^2 - su^2 - sv^2 > 0 \text{ for } s,$$

then

$$\sigma_s(S_{t,s}((a,b))) = x + i\sqrt{y^2 - su^2 - sv^2} \in (\mathbb{C} \setminus \mathbb{R}).$$

So, in such a case, the (t,s)-shift $S_{t,s}$ does not preserve the spectral property of $(a,b) \in \mathbb{H}_t$ to that of $(a,b) \in \mathbb{H}_s$.

As one can see in Propositions 21 and 25, indeed, even though the scale-shifts $\{S_{t_1,t_2}\}_{t_1,t_2 \in \mathbb{R}}$ are $\mathbb{R}$-Banach-algebra-isomorphisms "over $\mathbb{R}$," they do not preserve the invertibility and spectral-properties on the scaled hypercomplex rings $\{\mathbb{H}_t\}_{t \in \mathbb{R}}$ studied in Section 2.

Now, from the $\mathbb{R}$-Banach algebras $\{\mathbb{H}_t\}_{t \in \mathbb{R}}$, define a "pure-algebraic" $\mathbb{R}$-algebra $\mathscr{H}$ by

$$\mathscr{H} \overset{\text{def}}{=} \underset{t \in \mathbb{R}}{\oplus^a} \mathbb{H}_t \tag{33}$$

where $\oplus^a$ is the pure-algebraic direct product of $\mathbb{R}$-algebras. By the definition (33), every element T of $\mathscr{H}$ is expressed by

$$T = \underset{t \in \mathbb{R}}{\oplus} h_t \in \mathscr{H}, \text{ with } h_t \in \mathbb{H}_t, \ \forall t \in \mathbb{R}.$$

But, it means actually that there exists $N \in \mathbb{N}$, such that

$$T = \overset{N}{\underset{j=1}{\oplus}} h_{t_j} \text{ in } \mathscr{H}, \text{ with } h_{t_j} \in \mathbb{H}_{t_j}^{\times}, \ \forall j = 1, ..., N, \tag{34}$$

understood to be

$$T = \left(\overset{N}{\underset{j=1}{\oplus}} h_{t_j} \right) \oplus \left(\underset{s \in \mathbb{R} \setminus \{t_1, \dots, t_N\}}{\oplus} (0 + 0i + 0j_s + 0k_s) \right),$$

by the algebraic direct product $\oplus^a$. i.e., an element $T = \underset{t \in \mathbb{R}}{\oplus} h_t \in \mathscr{H}$ has only finitely-many "non-zero" direct summands by the pure-algebraic direct product $\oplus^a$. If

$$h_1 \in \mathbb{H}_{t_1} \text{ and } h_2 \in \mathbb{H}_{t_2} \text{ in } \mathscr{H},$$

then

$$h_1 + h_2 = \begin{cases} h_1 + h_2 \in \mathbb{H}_{t_1}, & \text{if } t_1 = t_2 \\ h_1 \oplus h_2 \in \mathbb{H}_{t_1} \oplus \mathbb{H}_{t_2} & \text{otherwise,} \end{cases} \tag{35}$$

$$h_1 h_2 = \begin{cases} h_1 h_2 \in \mathbb{H}_{t_1} & \text{if } t_1 = t_2 \\ O = \underset{t \in \mathbb{R}}{\oplus} 0 \in \mathscr{H} & \text{otherwise,} \end{cases}$$

in $\mathscr{H}$, by (32), because

$$h_j = h_j \oplus \left(\underset{t \in \mathbb{R} \setminus \{t_j\}}{\oplus} 0 \right) \in \mathscr{H}, \ \ \forall j = 1, 2.$$

On this $\mathbb{R}$-algbebra $\mathscr{H}$ of (31), for any $s \in \mathbb{R}$, define an operator S_s acting on $\mathscr{H}$ by

$$S_s \left(\underset{t \in \mathbb{R}}{\oplus} h_t \right) \overset{\text{def}}{=} \underset{t \in \mathbb{R}}{\oplus} S_{t,t+s} \left(h_t \right), \tag{36}$$

for all $\underset{t \in \mathbb{R}}{\oplus} h_t \in \mathscr{H}$, with $h_t \in \mathbb{H}_t$, for all $s \in \mathbb{R}$, where $S_{t,t+s}$ in the right-hand side of (34) are the $(t, t+s)$-shifts. So, the definition (34) actually means that

$$S_s \left(\underset{t \in \mathbb{R}}{\oplus} h_t \right) = S_s \left(\overset{N}{\underset{j=1}{\oplus}} h_{t_j} \right) \overset{\text{def}}{=} \overset{N}{\underset{j=1}{\oplus}} S_{t_j, t_j+s} \left(h_{t_j} \right),$$

for any fixed $s \in \mathbb{R}$, where S_{t_j, t_j+s} are the (t_j, t_j+s)-shifts, for all $j = 1, ..., N$.

Definition 10. The $\mathbb{R}$-algebra $\mathscr{H} = \underset{t \in \mathbb{R}}{\oplus^a} \mathbb{H}_t$ of (33) is called the (scaled-)hypercomplex $\mathbb{R}$-algebra. The function S_s of (36) on $\mathscr{H}$ is called the hypercomplex-shift operator by $s \in \mathbb{R}$ (in short, s-hypercomplex shift) on $\mathscr{H}$.

By the definition (36) of hypercomplex shifts $\{S_s\}_{s \in \mathbb{R}}$, one obtains the following result.

Theorem 12. *An s-hypercomplex shift S_s is a $\mathbb{R}$-algebra-isomorphism on $\mathscr{H}$, for all $s \in \mathbb{R}$, i.e.,*

$$\mathscr{H} \overset{alg}{=} S_s \left(\mathscr{H} \right), \quad \forall s \in \mathbb{R}, \tag{37}$$

where "$\overset{alg}{=}$" means "being $\mathbb{R}$-algebra-isomorphic to."

Proof. Let $\mathscr{H} = \underset{t \in \mathbb{R}}{\oplus^a} \mathbb{H}_t$ be the hypercomplex $\mathbb{R}$-algebra, which is the pure-algebraic direct product of the scaled hypercomplexes $\{\mathbb{H}_t\}_{t \in \mathbb{R}}$. For each direct summand $\mathbb{H}_t$ for $t \in \mathbb{R}$, the $(t, t+s)$-shift $S_{t,t+s} : \mathbb{H}_t \to \mathbb{H}_{t+s}$ is a $\mathbb{R}$-Banach-algebra-isomrophism, for any $s \in \mathbb{R}$. So, the function S_s is a bijective $\mathbb{R}$-linear transformation satisfying

$$S_s \left(r_1 T_1 + r_2 T_2 \right) = r_1 S_s \left(T_1 \right) + r_2 S_s \left(T_2 \right),$$

by (35), for all $r_1, r_2 \in \mathbb{R}$ and $T_1, T_2 \in \mathscr{H}$, and hence, it becomes a $\mathbb{R}$-vector-space isomorphism for $s \in \mathbb{R}$. Moreover,

$$S_s \left(T_1 T_2 \right) = S_s \left(T_1 \right) S_s \left(T_2 \right), \ \forall T_1, T_2 \in \mathscr{H},$$

by (35), implying that S_s is a bijective multiplicative $\mathbb{R}$-vector-space isomorphism, equivalently, a $\mathbb{R}$-algebra-isomrophism, for $s \in \mathbb{R}$. Therefore the relation (37) holds. □

By definition, one obtains the following result.

Proposition 13. *Let S_s be the s-hypercomplex shift (36) on the hypercomplex $\mathbb{R}$-algebra $\mathscr{H}$, for $s \in \mathbb{R}$. Then*

$$S_s^{-1} = S_{-s}, \quad \text{on} \quad \mathscr{H}. \tag{38}$$

Proof. Since all hypercomplex shifts $\{S_t\}_{t\in\mathbb{R}}$ are $\mathbb{R}$-algebra-isomorphisms on $\mathscr{H}$, their inverses $\left\{S_t^{-1}\right\}_{t\in\mathbb{R}}$ are well-defined on $\mathscr{H}$, too. Observe that, for any

$$T = \underset{t\in\mathbb{R}}{\oplus} h_t \overset{\text{let}}{=} \overset{N}{\underset{j=1}{\oplus}} h_{t_j} \in \mathscr{H},$$

in the sense of (32), we have

$$S_{-s}S_s\left(T\right) = S_{-s}\left(\overset{N}{\underset{j=1}{\oplus}} h_{t_j+s}\right) = \overset{N}{\underset{j=1}{\oplus}} h_{(t_j+s)-s} = T,$$

and

$$S_sS_{-s}\left(T\right) = S_s\left(\overset{N}{\underset{j=1}{\oplus}} h_{t_j-s}\right) = \overset{N}{\underset{j=1}{\oplus}} h_{(t_j-s)+s} = T,$$

by (33), where $h_{t+s} \overset{\text{denote}}{=} S_{t,t+s}\left(h_t\right)$, for all $t, s \in \mathbb{R}$, implying that $S_s^{-1} = S_{-s}$, for all $s \in \mathbb{R}$. Therefore, the invertibility (38) for $\{S_s\}_{s\in\mathbb{R}}$ holds true. □

By (38), one can conclude that

$$\{S_s\}_{s\in\mathbb{R}} = \{S_{-s}\}_{s\in\mathbb{R}} = \left\{S_s^{-1}\right\}_{s\in\mathbb{R}},$$

set-theoretically. This set-equalities motivate the following result.

Theorem 14. *Let $\mathscr{S} \overset{\text{denote}}{=} \{S_s\}_{s\in\mathbb{R}}$ be the collection of all hypercomplex shifts on the hypercomplex $\mathbb{R}$-algebra $\mathscr{H}$. Then the pair $(\mathscr{S}, \cdot)$ forms an abelian group satisfying*

$$(\mathscr{S}, \cdot) \overset{\text{group}}{=} (\mathbb{R}, +)\,, \text{ the time-flow, or the continuum,} \tag{39}$$

where the operation $(\cdot)$ on $\mathscr{S}$ is the isomorphism-product (or, the composition), and "$\overset{\text{group}}{=}$" means "being group-isomrophic to."

Proof. Let $\mathscr{S}$ be the set of all hypercomplex shifts, and supppose $(\cdot)$ is the isomorphism-product. Then, for any $S_{s_1}, S_{s_2} \in \mathscr{S}$, one has

$$S_{s_1}S_{s_2} = S_{s_1+s_2}, \quad \text{on} \quad \mathscr{H},$$

since

$$\left(S_{s_1}S_{s_2}\right)\left(\underset{t\in\mathbb{R}}{\oplus} h_t\right) = S_{s_1}\left(\underset{t\in\mathbb{R}}{\oplus} h_{t+s_2}\right) = \underset{t\in\mathbb{R}}{\oplus} h_{t+s_2+s_1} = S_{s_1+s_2}\left(\underset{t\in\mathbb{R}}{\oplus} h_t\right),$$

for all $\underset{t\in\mathbb{R}}{\oplus} h_t \in \mathscr{H}$ (understood to be (34)), with $h_t \in \mathbb{H}_t$, for all $s_1, s_2 \in \mathbb{R}$, where $h_{t+s} \overset{\text{denote}}{=} S_{t,t+s}\left(h_t\right)$, for all $t, s \in \mathbb{R}$. So,

$$\left(S_{s_1}S_{s_2}\right)S_{s_3} = S_{s_1+s_2+s_3} = S_{s_1}\left(S_{s_2}S_{s_3}\right),$$

on $\mathscr{H}$, for all $s_1, s_2, s_3 \in \mathbb{R}$. Also, this family $\mathscr{S}$ contains $S_0 \in \mathscr{S}$ such that

$$S_sS_0 = S_{s+0} = S_s = S_{0+s} = S_0S_s, \text{ on } \mathscr{H},$$

for all $s \in \mathbb{R}$. Recall that, by (38), every $S_s \in \mathscr{S}$ has its inverse $S_s^{-1} = S_{-s}$ in $\mathscr{S}$. Thus, the pair $(\mathscr{S}, \cdot)$ forms a group. Moreover,

$$S_{s_1}S_{s_2} = S_{s_1+s_2} = S_{s_2+s_1} = S_{s_2}S_{s_1}, \text{ on } \mathscr{H},$$

for all $s_1, s_2 \in \mathbb{R}$, implying that this group $(\mathscr{S}, \cdot)$ is an abelian.

Define now a function $\Psi : \mathscr{S} \rightarrow \mathbb{R}$ by

$$\Psi\left(S_s\right) = s, \quad \forall s \in \mathbb{R}.$$

Then it is a well-defined bijection from $\mathscr{S}$ onto $\mathbb{R}$, satisfying

$$\Psi\left(S_{s_1}S_{s_2}\right) = \Psi\left(S_{s_1+s_2}\right) = s_1 + s_2 = \Psi\left(S_{s_1}\right) + \Psi\left(S_{s_2}\right),$$

for all $s_1, s_2 \in \mathbb{R}$. So, this bijection Ψ is a group-homormophism, and hence, it is a group-isomrophism. Therefore, the groups $(\mathscr{S}, \cdot)$ and $(\mathbb{R}, +)$ are isomorphic. $\square$

By (39), the family $\mathscr{S} = \{S_s\}_{s\in\mathbb{R}}$ of the hypercomplex shifts forms an abelian group isomorphic to the time-flow $(\mathbb{R}, +)$. It means that the family $\mathscr{S}$ provides a classical dynamics on the system $\mathscr{H}$, the direct product of the scaled-hypercomplexes $\{\mathbb{H}_t\}_{t\in\mathbb{R}}$ up to (37).

In the first part of this section, we showed that even though our (t, s)-shift $S_{t,s}$ is a $\mathbb{R}$-Banach-algebra-isomrophism from $\mathbb{H}_t$ onto $\mathbb{H}_s$, it does not preserve the invertibility and the spectral properties on the scaled hypercomplexes, considered in Section 2. Similarly, one can verify that all elements of $\mathscr{S}$ does not preserve the invertibility and spectral properties of Section 2 on the direct summands $\{\mathbb{H}_t\}_{t\in\mathbb{R}}$, in general, inside the hypercomplex $\mathbb{R}$-algebra $\mathscr{H}$. From below, we fix

$$T = \underset{j=1}{\overset{N}{\oplus}} h_{t_j} \in \mathscr{H}, \text{ with } h_{t_j} \in \mathbb{H}_{t_j}^{\times}, \ \forall j = 1, ..., N, \tag{40}$$

for $N \in \mathbb{N}$.

Lemma 5. Let $T \in \mathscr{H}$ be in the sense of (40). Then T is invertible in the subalgebra $\underset{j=1}{\overset{N}{\oplus}}{}^{a}\mathbb{H}_{t_j}$ of $\mathscr{H}$, if and only if the direct summands h_{t_j} are invertible in $\mathbb{H}_{t_j}$, for all $j = 1, ..., N$. i.e.,

$$T \text{ is invertible in } \underset{j=1}{\overset{N}{\oplus}}{}^{a}\, \mathbb{H}_{t_j} \Longleftrightarrow h_{t_j} \text{ are invertible in } \mathbb{H}_{t_j}, \forall j = 1, ..., N \tag{41}$$

Proof. Let H_j be Hilbert spaces (over $\mathbb{C}$), and $B\left(H_j\right)$, the corresponding operator algebras, for $j = 1, ..., N$, for $N \in \mathbb{N}$, and suppose A_j are the C^*-subalgebras of $B\left(H_j\right)$, for all $j = 1, ..., N$. Then it is well-known that

$$\underset{j=1}{\overset{N}{\oplus}} T_j \text{ is invertible in } \underset{j=1}{\overset{N}{\oplus}} A_j \Longleftrightarrow T_j \text{ are invertible in } A_j, \ \forall j = 1, ..., N.$$

(e.g., see [22]). Under our canonical Hilbert-space representations $\left(\mathbb{C}^2, \pi_{t_j}\right)$ of the t_j-scaled hypercomplex ring $\mathbb{H}_{t_j}$, realized to be the Hilbert-space operators in $\mathcal{H}_2^{t_j}$ (in $M_2(\mathbb{C})$), we have

$$T \text{ is invertible in } \underset{j=1}{\overset{N}{\oplus}}{}^{a}\, \mathbb{H}_j \Longleftrightarrow h_{t_j} \text{ are invertible in } \mathbb{H}_t, \ \forall j.$$

Therefore, the characterization (41) is obtained. $\square$

Note that, the invertibility of T of (41) is considerd on the "subalgebra $\underset{j=1}{\overset{N}{\oplus}}{}^{a}\mathbb{H}_{t_j}$" in the hypercomplex $\mathbb{R}$-algebra $\mathscr{H}$ (not wholly on $\mathscr{H}$) because

$$T = \underset{j=1}{\overset{N}{\oplus}} h_{t_j} = \left(\underset{j=1}{\overset{N}{\oplus}} h_{t_j}\right) + \left(\underset{t\in\mathbb{R}\setminus\{t_1,...,t_N\}}{\oplus} (0 + 0i + 0j_t + 0k_t)\right)$$

is clearly not invertible (in any senses over $\mathbb{R}$, or over $\mathbb{C}$) in $\mathscr{H}$.

Theorem 15. *Let S_s be the s-hypercomplex shift on the hypercomplex $\mathbb{R}$-algebra $\mathscr{H}$, and let $T \in \mathscr{H}$ be in the sense of (40). Then*

$$S_s(T) \text{ is invertible in } \overset{N}{\underset{j=1}{\oplus^a}} \mathbb{H}_{t_j+s}, \Longleftrightarrow S_s\left(h_{t_j}\right) \in \mathbb{H}_{t_j} \text{ is invertible}, \forall j \tag{42}$$

Proof. The relation (42) holds by (41). □

The above theorem characterizes the invertibility on certain subalgebras inside $\mathscr{H}$ by the invertibility on $\{\mathbb{H}_t\}_{t\in\mathbb{R}}$ (over $\mathbb{C}$).

Lemma 6. Let $T \in \mathscr{H}$ be in the sense of (40). Define the spectrum spec(T) of T by

$$\text{spec}(T) \overset{\text{def}}{=} \text{spec}\left(\overset{N}{\underset{j=1}{\oplus}} \left[h_{t_j}\right]_{t_j}\right). \tag{43}$$

Then

$$\text{spec}(T) = \overset{N}{\underset{j=1}{\cup}} \left\{\sigma_{t_j}\left(h_{t_j}\right), \overline{\sigma_{t_j}\left(h_{t_j}\right)}\right\}, \text{ in } \mathbb{C} \tag{44}$$

where σ_{t_j} are the t_j-spectralizations, for all $j = 1, ..., N$.

Proof. Let T be in the sense of (40) with its non-zero direct summands $h_{t_j} \in \mathbb{H}_{t_j}^{\times}$ in $\mathbb{H}_{t_j}$, for $j = 1, ..., N$. Then all elements of the subalgebra $\overset{N}{\underset{j=1}{\oplus^a}} \mathbb{H}_{t_j}$ of the hypercomplex $\mathbb{R}$-algebra $\mathscr{H}$ are acting on $\left(\mathbb{C}^2\right)^{\oplus N} = \underbrace{\mathbb{C}^2 \oplus ... \oplus \mathbb{C}^2}_{N\text{-times}}$, because each direct summands $\mathbb{H}_{t_l}$ of $\overset{N}{\underset{j=1}{\oplus^a}} \mathbb{H}_{t_j}$ has the canonical representation $\left(\mathbb{C}^2, \pi_{t_l}\right)$, for all $l = 1, ..., N$. i.e., this subalgebra has a Hilbert-space representation,

$$\left(\left(\mathbb{C}^2\right)^{\oplus N}, \pi \overset{\text{denote}}{=} \overset{N}{\underset{j=1}{\oplus}} \pi_{t_j}\right),$$

over $\mathbb{C}$. Under this representation, if T is as above, then it is realized to be

$$\pi(T) = \overset{N}{\underset{j=1}{\oplus}} \pi_{t_j}\left(h_{t_j}\right) = \overset{N}{\underset{j=1}{\oplus}} \left[h_{t_j}\right]_{t_j},$$

in $\pi\left(\overset{N}{\underset{j=1}{\oplus^a}} \mathbb{H}_{t_j}\right) = \overset{N}{\underset{j=1}{\oplus^a}} \mathcal{H}_2^{t_j}$, contained in $\left(M_2(\mathbb{C})\right)^{\oplus N}$. So, the spectrum spec$(T)$ of (43) is well-defined, i.e.,

$$\text{spec}(T) \overset{\text{def}}{=} \text{spec}\left(\overset{N}{\underset{j=1}{\oplus}} \left[h_{t_j}\right]_{t_j}\right),$$

in the operator algebra $B\left(\mathbb{C}^{2N}\right)$.

Suppose $T_l \in B(H_l)$ are operators on Hilbert spaces H_l, for $l = 1, 2$. It is well-known that if

$$T_1 \oplus T_2 \in B(H_1) \oplus B(H_2) = B(H_1 \oplus H_2),$$

then

$$\text{spec}(T_1 \oplus T_2) = \text{spec}(T_1) \cup \text{spec}(T_2),$$

in $\mathbb{C}$ (e.g., see [21] and [22]). Therefore, if spec(T) is defined as in (43), then

$$\text{spec}(T) = \overset{N}{\underset{j=1}{\cup}} \text{spec}\left(\left[h_{t_j}\right]_{t_j}\right), \text{ in } \mathbb{C}.$$

Since

$$\text{spec}\left(\left[h_{t_j}\right]_{t_j}\right) = \left\{\sigma_{t_j}\left(h_{t_j}\right), \overline{\sigma_{t_j}\left(h_{t_j}\right)}\right\},$$

for all $j = 1, ..., N$, the set-equality (44) holds. □

By the relation (44) induced by the definition (43), we have the following result.

Theorem 16. *Let $T \in \mathscr{H}$ be in the sense of (40), and let $S_s \in \mathscr{S}$ be the s-hypercomplex shift on $\mathscr{H}$. Then* $\text{spec}\left(S_s\left(T\right)\right)$ *is well-defined as in (43), and*

$$\text{spec}\left(S_s\left(T\right)\right) = \overset{N}{\underset{j=1}{\cup}} \left\{\sigma_{t_j+s}\left(S_{t_j,t_j+s}\left(h_{t_j}\right)\right), \overline{\sigma_{t_j+s}\left(S_{t_j,t_j+s}\left(h_{t_j}\right)\right)}\right\} \tag{45}$$

in $\mathbb{C}$, where σ_{t_j+s} are the (t_j+s)-spectralizations, for all $j = 1, ..., N$.

Proof. The set-equality (45) is obtained by (44) under the acton of $S_s \in \mathscr{S}$. □

In this section, we studied the system $\mathscr{H}$, the hypercomplex $\mathbb{R}$-algebra, of scaled hypercomplexes $\{\mathbb{H}_t\}$, and certain $\mathbb{R}$-algebra-isomorphisms $\mathscr{S} = \{S_s\}_{s\in\mathbb{R}}$ on $\mathscr{H}$, inducing a trivial dynamics on $\{\mathbb{H}_t\}_{t\in\mathbb{R}}$. Unfortunately, the $\mathbb{R}$-algebra-isomorphisms of $\mathscr{S}$ preserve neither the invertibility nor the spectral properties on $\{\mathbb{H}_t\}_{t\in\mathbb{R}}$ inside $\mathscr{H}$ in general, however, at least, we observed why they are not preserved by $\mathscr{S}$, and how they are understood under the action of $\mathscr{S}$.

4. Certain Analytic Data on $\mathbb{H}_t$ Depending on a Scale $t \in \mathbb{R}$

In this section, we focus on each t-scaled hypercomplexes $\mathbb{H}_t$ for a scale $t \in \mathbb{R}$, and study certain analytic data on $\mathbb{H}_t$ in terms of a natural linear functional over $\mathbb{R}$ (in short, a $\mathbb{R}$-linear functional). In particular, we are interested in a $\mathbb{R}$-linear functional τ_t on $\mathbb{H}_t$ induced by the normalized trace $\tau = \frac{1}{2}tr$ on $M_2(\mathbb{C})$, where tr is the usual trace on $M_2(\mathbb{C})$ "over $\mathbb{C}$." Since the t-scaled hypercomplexes $\mathbb{H}_t$ is regarded as its realization $\mathcal{H}_2^t = \pi_t(\mathbb{H}_t)$ up to its representation $\left(\mathbb{C}^2, \pi_t\right)$, the normalized trace τ on $M_2(\mathbb{C})$ is restricted to be the $\mathbb{R}$-linear functional $\tau\mid_{\mathcal{H}_2^t}$. Define a $\mathbb{R}$-linear functional τ_t on $\mathbb{H}_t$ by

$$\tau_t \overset{\text{def}}{=} \tau \circ \pi_t : \mathbb{H}_t \to \mathbb{R}, \tag{46}$$

where π_t is the action of $\mathbb{H}_t$, and τ is the normalized trace on $M_2(\mathbb{C})$. Note that the restriction τ_t of the $\mathbb{C}$-trace τ becomes a $\mathbb{R}$-trace on $\mathbb{H}_t$, because

$$\tau_t\left((a,b)\right) = \tau\left([(a,b)]_t\right) = \tau\left(\begin{pmatrix} a & tb \\ \overline{b} & \overline{a} \end{pmatrix}\right) = \frac{1}{2}\left(a + \overline{a}\right), \tag{47}$$

i.e.,

$$\tau_t\left((a,b)\right) = \frac{1}{2}\left(a + \overline{a}\right) = \text{Re}_{\mathbb{C}}\left(a\right),$$

where $\text{Re}_{\mathbb{C}}(\bullet)$ is the real part on $\mathbb{C}$.

By understanding $\mathbb{H}_t$ as $\text{span}_{\mathbb{R}}\{1, i, j_t, k_t\}$, one can define the real part $\text{Re}(\bullet)$ and the imaginary part $\text{Im}(\bullet)$ on $\mathbb{H}_t$ by

$$\text{Re}\left(x + yi + uj_t + vk_t\right) = x, \tag{48}$$

and

$$\text{Im}\left(x + yi + uj_t + vk_t\right) = yi + uj_t + vk_t,$$

for all $x, y, u, v \in \mathbb{R}$.

Proposition 17. *The $\mathbb{R}$-linear functional τ_t of (46) is identified with the real part* $\text{Re}(\bullet)$ *of (48) on $\mathbb{H}_t$. i.e.,*

$$\tau_t\left(h\right) = \text{Re}\left(h\right), \quad \forall h \in \mathbb{H}_t. \tag{49}$$

Proof. We have $\tau_t = \text{Re}\,(\bullet)$ on $\mathbb{H}_t$, by (46) and (47). Indeed,

$$\tau_t\,(x + yi + uj_t + vk_t) = \tau_t\,((x + yi,\ u + vi)),$$

identical to

$$\text{Re}_{\mathbb{C}}\,(x + yi) = x = \text{Re}\,(x + yi + uj_t + vk_t),$$

for all $x + yi + uj_t + vk_t \in \mathbb{H}_t$ with $x, y, u, v \in \mathbb{R}$. □

By the above proposition, one can identify the $\mathbb{R}$-trace τ_t with the real part $\text{Re}\,(\bullet)$ on $\mathbb{H}_t$ by (49). Then the $\mathbb{R}$-basis elements $\{1, i, j_t, k_t\}$ of $\mathbb{H}_t$ satisfy the following analytic data in terms of the $\mathbb{R}$-trace τ_t of (46).

Theorem 18. *If $\tau_t = \text{Re}\,(\bullet)$ is the $\mathbb{R}$-trace (46), then*

$$(\tau_t\,(1^n))_{n=1}^{\infty} = \left(\underline{1, 1, 1, 1}, 1, 1, 1, 1, ...\right); \tag{50}$$

and

$$(\tau_t\,(i^n)) = \left(\underline{0, -1, 0, 1}, 0, -1, 0, 1, ...\right),$$

where the symbol "$\underline{r_1, r_2, r_3, r_4}$" means the first four entries repeatedly, or periodically appeared in a sequence $(r_n)_{n=1}^{\infty}$; and

$$\tau_t\,(j_t^n) = \tau_t\,(k_t^n) = \begin{cases} 0 & \text{if } n \in 2\mathbb{N} - 1 \\ t^{\frac{n}{2}} & \text{if } n \in 2\mathbb{N}, \end{cases} \tag{51}$$

for all $n \in \mathbb{N}$, where $kY = \{ky : y \in Y\}$ and $Y \pm l = \{y \pm l : y \in Y\}$, for all subsets Y of $\mathbb{N}$, and $k, l \in \mathbb{N}$.

Proof. Clearly, one has $1^n = 1 = 1 + 0i + 0j_t + 0k_t$ in $\mathbb{H}_t$, for all $n \in \mathbb{N}$, since 1 is the unity of $\mathbb{H}_t$, and hence,

$$\tau_t\,(1^n) = \tau_t\,(1) = \text{Re}\,(1) = 1, \quad \forall n \in \mathbb{N}.$$

Also, we have

$$i^n = \begin{cases} \pm i & \text{if } n \in 2\mathbb{N} - 1 \\ -1 & \text{if } n \in 2\mathbb{N} \setminus 4\mathbb{N} \\ 1 & \text{if } n \in 4\mathbb{N}, \end{cases}$$

and hence,

$$\tau_t\,(i^n) = \text{Re}\,(i^n) = \begin{cases} 0 & \text{if } n \in 2\mathbb{N} - 1 \\ -1 & \text{if } n \in 2\mathbb{N} \setminus 4\mathbb{N} \\ 1 & \text{if } n \in 4\mathbb{N}, \end{cases}$$

for all $n \in \mathbb{N}$. Thus the analytic-data sequences of (50) are obtained.

Observe that $j_t^2 = t$, $j_t^3 = j_t^2 j_t = t j_t$, $j_t^4 = t^2$, and $j_t^5 = t^2 j_t$, etc.. So, inductively, we have

$$j_t^n = \begin{cases} t^{\frac{n-1}{2}} j_t & \text{if } n \in 2\mathbb{N} - 1 \\ t^{\frac{n}{2}} & \text{if } n \in 2\mathbb{N}, \end{cases}$$

for all $n \in \mathbb{N}$. Similarly, one obtains that

$$k_t^n = \begin{cases} t^{\frac{n-1}{2}} j_t & \text{if } n \in 2\mathbb{N} - 1 \\ t^{\frac{n}{2}} & \text{if } n \in 2\mathbb{N}, \end{cases}$$

for all $n \in \mathbb{N}$. Therefore,

$$\text{Re}\,(j_t^n) = \text{Re}\,(k_t^n) = \begin{cases} 0 & \text{if } n \in 2\mathbb{N} - 1 \\ t^{\frac{n}{2}} & \text{if } n \in 2\mathbb{N}, \end{cases}$$

for all $n \in \mathbb{N}$. It implies the analytic data (51). □

The above theorem fully characterizes the analytic data of the $\mathbb{R}$-basis elements $\{1, i, j_t, k_t\}$ in terms of the $\mathbb{R}$-trace τ_t on $\mathbb{H}_t$, by (50) and (51). Especially, by (51), on the 0-scaled hypercomplexes $\mathbb{H}_0$, the $\mathbb{R}$-basis elements j_0 and k_0 have the 0-analytic data in the sense that

$$\left(\tau_0\left(j_0^n\right)\right)_{n=1}^{\infty} = \left(\tau_0\left(k_0^n\right)\right)_{n=1}^{\infty} = \left(\underline{0, 0, 0, 0}, 0, 0, ...\right).$$

Let $\mathscr{H} = \underset{t \in \mathbb{R}}{\oplus^a} \mathbb{H}_t$ be the hypercomplex $\mathbb{R}$-algebra (33). Then one can define a "unbounded" $\mathbb{R}$-linear functional $\varphi : \mathscr{H} \to \mathbb{C}$ by

$$\varphi \overset{\text{def}}{=} \underset{t \in \mathbb{R}}{\oplus} \tau_t, \quad \text{on} \quad \mathscr{H}, \tag{52}$$

i.e.,

$$\varphi\left(\overset{N}{\underset{l=1}{\oplus}} h_{t_l}\right) = \sum_{l=1}^{N} \tau_{t_l}\left(h_{t_l}\right), \quad \forall \overset{N}{\underset{l=1}{\oplus}} h_{t_l} \in \mathscr{H}.$$

Since every element $T \in \mathscr{H}$ is a "finite" direct sum in $\underset{t \in \mathbb{R}}{\cup} \mathbb{H}_t$, the above morphism φ of (52) is well-defined on $\mathscr{H}$, as a "unbounded" linear functional over $\mathbb{R}$. Even though it is unbounded, it is strongly bounded in the sense that: for each "fixed" $T \in \mathscr{H}$, $|\varphi(T)| < \infty$, because (i) T is a finite direct sum, and (ii) $\{\tau_t\}_{t \in \mathbb{R}}$ are bounded on $\{\mathbb{H}_t\}_{t \in \mathbb{R}}$, respectively.

5. On the hypercomplex $\mathbb{R}$-Algebra $\mathscr{H}$

Let $\mathscr{H}$ be the hypercomplex $\mathbb{R}$-algebra (33), and let $\mathscr{S} = \{S_s\}_{s \in \mathbb{R}}$ be the group (39) of all hypercomplex shifts on $\mathscr{H}$ satisfying (38). At the end of Section 4, we defined a $\mathbb{R}$-linear functional (52),

$$\varphi : \mathscr{H} \to \mathbb{R}, \tag{53}$$

by

$$\varphi\left(\underset{t \in \mathbb{R}}{\oplus} h_t\right) \overset{\text{def}}{=} \sum_{t \in \mathbb{R}} \tau_t\left(h_t\right) = \sum_{t \in \mathbb{R}} \text{Re}\left(h_t\right),$$

where $\tau_t = \text{Re}(\bullet)$ are the $\mathbb{R}$-traces (46), for all $t \in \mathbb{R}$. For example, if t_1, t_2, t_3, t_4 are mutually distinct in $\mathbb{R} \setminus \{0\}$, and

$$h = j_{t_1} \oplus k_{t_2} \oplus i_{t_3} \oplus j_{t_4} \in \mathscr{H},$$

where $j_{t_1} \in \mathbb{H}_{t_1}$, $k_{t_2} \in \mathbb{H}_{t_2}$, $i_{t_3} = i \in \mathbb{H}_{t_3}$, and $j_{t_4} \in \mathbb{H}_{t_4}$ in $\mathscr{H}$, then

$$h^n = j_{t_1}^n \oplus k_{t_2}^n \oplus i_{t_3}^n \oplus j_{t_4}^n \in \mathscr{H}, \ \forall n \in \mathbb{N},$$

and hence,

$$\varphi\left(h^n\right) = \tau_{t_1}\left(j_{t_1}^n\right) + \tau_{t_2}\left(k_{t_2}^n\right) + \tau_{t_3}\left(i^n\right) + \tau_{t_4}\left(j_{t_4}^n\right),$$

identical to

$$\varphi\left(h^n\right) = \text{Re}\left(j_{t_1}^n\right) + \text{Re}\left(k_{t_2}^n\right) + \text{Re}\left(i^n\right) + \text{Re}\left(j_{t_4}^n\right),$$

in $\mathbb{R}$, for all $n \in \mathbb{R}$, where

$$\text{Re}\left(j_{t_1}^n\right) = \text{Re}\left(k_{t_2}^n\right) = \text{Re}\left(j_{t_4}^n\right) = \begin{cases} 0 & \text{if } n \in 2\mathbb{N} - 1 \\ t^{\frac{n}{2}} & \text{if } n \in 2\mathbb{N}, \end{cases}$$

and

$$\text{Re}\left(i^n\right) = \begin{cases} 0 & \text{if } n \in 2\mathbb{N} - 1 \\ -1 & \text{if } n \in 2\mathbb{N} \setminus 4\mathbb{N} \\ 1 & \text{if } n \in 4\mathbb{N}, \end{cases}$$

for all $n \in \mathbb{N}$, by (50) and (51).

5.1. Analytic Data on $\mathscr{H}$ Deformed by the Action of $(\mathscr{S}, \cdot)$

In this section, we study analytic data on the hypercomplex $\mathbb{R}$-algebra $\mathscr{H} = \underset{t\in\mathbb{R}}{\oplus}^{a}\mathbb{H}_t$, with respect to the $\mathbb{R}$-trace $\varphi = \underset{t\in\mathbb{R}}{\oplus}\tau_t$ of (53), and let

$$\mathscr{S} = \{S_s : s \in \mathbb{R}\}$$

be the family of s-hypercomplex shifts (37) on $\mathscr{H}$, inducing the time-flow $(\mathscr{S}, \cdot)$ on $\mathscr{H}$, by (39). Even though each s-shift $S_s \in \mathscr{S}$ is a $\mathbb{R}$-algebra-isomorphisms on $\mathscr{H}$, one may verify that the action of $\mathscr{S}$ on $\mathscr{H}$ deforms the φ-depending analytic data on $\mathscr{H}$.

Observe that, since $S_s \in \mathscr{S}$ is a $\mathbb{R}$-algebra-isomorphism on $\mathscr{H}$, assigning,

$$S_s\left(\underset{j=1}{\overset{N}{\oplus}} h_{t_j}\right) = \underset{j=1}{\overset{N}{\oplus}} h_{t_j+s} \overset{\text{denote}}{=} \underset{j=1}{\overset{N}{\oplus}} S_{t_j,t_j+s}\left(h_{t_j}\right),$$

for any $h_t \in \mathbb{H}_t$ in $\mathscr{H}$, and $N \in \mathbb{N}$, for all $t \in \mathbb{R}$, one has

$$S_s(w_t) = w_{t+s} \in \{i_{t+s} = i, j_{t+s}, k_{t+s}\} \in \mathscr{H},$$

for all $w_t \in \{i_t = i, j_t, k_t\} \in \mathbb{H}_t$, implying that

$$\varphi\left((S_s(w_t))^n\right) = \varphi\left(w_{t+s}^n\right) \neq \varphi\left(w_t^n\right), \text{ for } n \in \mathbb{N},$$

since

$$\tau_{t+s}\left(w_{t+s}^n\right) = \mathrm{Re}\left(w_{t+s}^n\right) \neq \mathrm{Re}\left(w_t^n\right) = \tau_t\left(w_t^n\right),$$

for $n \in \mathbb{N}$, in general, by (50) and (51).

Lemma 7. Let $t \in \mathbb{R}$, and let 1, $i_t = i$, j_t, k_t be the $\mathbb{R}$-basis elements of $\mathbb{H}_t$ in $\mathscr{H}$. If $S_s \in \mathscr{S}$, then

$$\left(\varphi\left(S_s(1)^n\right)\right)_{n=1}^{\infty} = \left(\underline{1,1,1,1},1,1,1,1,...\right), \tag{54}$$

$$\left(\varphi\left(S_s(i_t)^n\right)\right)_{n=1}^{\infty} = \left(\underline{0,-1,0,1},0,-1,0,1,...\right) = \left(\varphi\left(i_t^n\right)\right)_{n=1}^{\infty}, \tag{55}$$

and

$$\varphi\left(S_s(j_t)^n\right) = \varphi\left(S_s(k_t)^n\right) = \begin{cases} 0 & \text{if } n \in 2\mathbb{N}-1 \\ (t+s)^{\frac{n}{2}} & \text{if } n \in 2\mathbb{N}, \end{cases} \tag{56}$$

for all $n \in \mathbb{N}$, for all $s \in \mathbb{R}$.

Proof. Consider that

$$S_s(1) = 1 \in \mathbb{H}_{t+s}, \quad \text{in} \quad \mathscr{H},$$

and

$$S_s(i_t) = i_{t+s} = i \in \mathbb{H}_{t+s}, \quad \text{in} \quad \mathscr{H},$$

for all $s, t \in \mathbb{R}$, satisfying

$$\left(\varphi\left(S_s(1)^n\right)\right)_{n=1}^{\infty} = \left(\tau_{t+s}\left(1^n\right)\right)_{n=1}^{\infty} = \left(\mathrm{Re}\left(1^n\right)\right)_{n=1}^{\infty},$$

respectively,

$$\left(\varphi\left(S_s(i_t)^n\right)\right)_{n=1}^{\infty} = \left(\tau_{t+s}\left(i_{t+s}^n\right)\right)_{n=1}^{\infty} = \left(\mathrm{Re}\left(i_t^n\right)\right)_{n=1}^{\infty},$$

Therefore, the relations (54) and (55) hold by (50), for all $t, s \in \mathbb{R}$.

Similarly, since

$$\left(\varphi\left(S_s(j_t)^n\right)\right)_{n=1}^{\infty} = \left(\tau_{t+s}\left(j_{t+s}^n\right)\right)_{n=1}^{\infty} = \left(\tau_{t+s}\left(k_{t+s}^n\right)\right)_{n=1}^{\infty} = \left(\varphi\left(S_s(k_t)^n\right)\right)_{n=1}^{\infty},$$

the analytic data (56) is obtained by (51), for the replaced scale $t+s \in \mathbb{R}$. ☐

By the above lemma, we obtain the following result.

Theorem 19. *For all* $T = \underset{l=1}{\overset{N}{\oplus}} h_{t_l} \in \mathscr{H}$ *with* $h_{t_l} \in \mathbb{H}_{t_l}^{\times}$, *for* $l = 1, ..., N$, *for all* $N \in \mathbb{N}$,

$$\left(\varphi\left(S_s\left(T\right)^n\right)\right)_{n=1}^{\infty} = \left(\varphi\left(T^n\right)\right)_{n=1}^{\infty} \text{ as } \mathbb{R}\text{-sequences,} \tag{57}$$

if and only if

$$s = 0, \quad \text{in} \quad \mathbb{R}.$$

Proof. By definition, for any $s \in \mathbb{R}$, since S_s is a $\mathbb{R}$-algebra isomorphism on $\mathscr{H}$, we have

$$S_s\left(T\right)^n = S_s\left(T^n\right) = S_s\left(\underset{l=1}{\overset{N}{\oplus}} h_{t_l}^n\right) = \underset{l=1}{\overset{N}{\oplus}} h_{t_l+s}^n \in \mathscr{H},$$

with $h_{t_l+s} = S_{t_j, t_j+s}\left(h_{t_l}\right)$, for all $l = 1, ..., N$, for all $n \in \mathbb{N}$, and

$$\varphi\left(S_s\left(h\right)^n\right) = \sum_{l=1}^{N} \text{Re}\left(h_{t_l+s}^n\right) \in \mathbb{R}.$$

So, to find the characterization of the equalities $\varphi\left(S_s\left(T\right)^n\right) = \varphi\left(T^n\right)$ for all $n \in \mathbb{N}$, for "all" such $T \in \mathscr{H}$, it suffices to show that

$$\tau_{t_l+s}\left(h_{t_l}^n\right) = \tau_{t_l+s}\left(S_s\left(h_{t_l}\right)^n\right) = \tau_t\left(h_{t_l}^n\right), \quad \forall l = 1, ..., N.$$

Now, fix $l \in \{1, ..., N\}$, and

$$h_{t_l} = x + yi_{t_l} + uj_{t_l} + vk_{t_l} \in \mathbb{H}_{t_l}, \text{ in } \mathscr{H},$$

with $x, y, u, v \in \mathbb{R}$. Then

$$S_s\left(h_{t_l}\right) = x + yi_{t_l+s} + uj_{t_l+s} + vk_{t_l+s} \in \mathbb{H}_{t_j+s},$$

in $\mathscr{H}$.

Recall and note that, we have

$$\left(\tau_t\left(i_t^n\right)\right)_{n=1}^{\infty} = \left(\underline{0, -1, 0, 1}, 0, -1, 0, 1, ...\right), \tag{58}$$

and

$$\left(\tau_t\left(\zeta_t^n\right)\right)_{n=1}^{\infty} = \left(\tau_t\left(\kappa_t^n\right)\right)_{n=1}^{\infty} = \left(0, t, 0, t^2, 0, t^3, 0, t^4, 0, ...\right),$$

by (50) and (51). So, if $s = 0$, then

$$\tau_t\left(h_{t_l}^n\right) = \tau_{t+0}\left(S_0\left(h_{t_l}\right)^n\right), \quad \forall n \in \mathbb{N},$$

by (58). Conversely, let's assume that

$$\tau_t\left(h_t^n\right) = \tau_{t+s}\left(h_{t+s}^n\right) = \tau_{t+s}\left(S_s\left(h_t^n\right)\right), \quad \forall n \in \mathbb{N},$$

and

$$s \neq 0 \quad \text{in} \quad \mathbb{R}.$$

Then, by (50) and (51), in particular, by (51),

$$\tau_{t+s}\left(h_{t_l+s}^n\right) \neq \tau_t\left(h_{t_l}^n\right), \text{ in general,}$$

by (58), contradicting our assumption. Therefore, $\varphi\left(S_s\left(T\right)^n\right) = \varphi\left(T^n\right)$ in $\mathscr{H}$, for all $T \in \mathscr{H}$, and $n \in \mathbb{N}$, if and only if $s = 0$ in $\mathbb{R}$. So, the relation (57) holds. □

The above characterization (57) seems natural, but it illustrates that the only 0-hypercomplex shift S_0, which is the group-identity of $\mathscr{S}$, can preserve analytic data on $\mathscr{H}$ up to the $\mathbb{R}$-trace φ. Equivalently, the analytic data on $\mathscr{H}$ up to the $\mathbb{R}$-linear functional φ are distorted by the action of $\mathscr{S} \setminus \{S_0\}$. So, it is interesting enough to consider how they are deformed where $s \to \infty$, or $s \to -\infty$, for the shifts $\{S_s\}_{s \in \mathbb{R}}$, by applying (54), (55) and (56).

5.2. Some Asymptotic Analytic Data on $\mathscr{H}$ under the Action of $(\mathscr{S}, \cdot)$

In this section, we focus on studying how certain asymptotic action of the group $(\mathscr{S}, \cdot)$ affect the analytic data on the hypercomplex $\mathbb{R}$-algebra $\mathscr{H} \overset{\text{def}}{=} \underset{t \in \mathbb{R}}{\oplus}^{a} \mathbb{H}_t$, up to the $\mathbb{R}$-linear functional $\varphi = \underset{t \in \mathbb{R}}{\oplus} \tau_t$ of (53). In other words, we are interested in the cases where we take s-hypercomplex shifts $S_s \in \mathscr{S}$, where either $s \to \infty$, or $s \to -\infty$ in $\mathbb{R}$, equivalently, where $|s|$ is "suitably" big enough in $\mathbb{R}$.

Recall that, for any t-scaled hypercomplexes,

$$\mathbb{H}_t = \text{span}_{\mathbb{R}} \left\{ 1_t \overset{\text{denote}}{=} 1,\ i_t \overset{\text{denote}}{=} i,\ j_t,\ k_t \right\},$$

as a direct summand of $\mathscr{H}$, one obtains the following analytic data up to the $\mathbb{R}$-linear functional φ on $\mathscr{H}$; and

$$\begin{aligned} \left(\varphi\left(1_t^n\right)\right)_{n=1}^{\infty} &= \left(\underline{1,1,1,1},1,1,1,1,\ldots\right), \\ \left(\varphi\left(i_t^n\right)\right)_{n=1}^{\infty} &= \left(\underline{0,-1,0,1},0,-1,0,1,\ldots\right), \\ \varphi\left(j_t^n\right) = \varphi\left(k_t^n\right) &= \begin{cases} 0 & \text{if } n \in 2\mathbb{N}-1 \\ t^{\frac{n}{2}} & \text{if } n \in 2\mathbb{N}, \end{cases} \end{aligned} \tag{59}$$

for all $n \in \mathbb{N}$, by (50) and (51). These data (59) provide the building blocks for computing the analytic data on $\mathscr{H}$ up to φ (Also, see Section 6 below).

Theorem 20. *Let $\{1_t = 1, i_t = i, j_t, k_t\}$ be the $\mathbb{R}$-basis of the t-scaled hypercomplexes $\mathbb{H}_t$, as a direct summand of the hypercomplex $\mathbb{R}$-algebra $\mathscr{H}$, for $t \in \mathbb{R}$, and let $\mathscr{S} = \{S_s\}_{s \in \mathbb{R}}$ be the family of all hypercomplex shifts on $\mathscr{H}$. Then*

$$\left(\varphi\left(\left(\lim_{s\to\infty} S_s\left(1_t\right)\right)^n\right)\right)_{n=1}^{\infty} = \left(\underline{1,1,1,1},1,1,\ldots\right) \tag{60}$$

$$\left(\varphi\left(\left(\lim_{s\to\infty} S_s\left(i_t\right)\right)^n\right)\right)_{n=1}^{\infty} = \left(\underline{0,-1,0,1},0,-1,0,1,\ldots\right) \tag{61}$$

and

$$\varphi\left(\left(\lim_{s\to\infty} S_s\left(w_t\right)\right)^n\right) = \begin{cases} 0 & \text{if } n \in 2\mathbb{N}-1 \\ \infty & \text{if } n \in 2\mathbb{N}, \end{cases} \tag{62}$$

for all $n \in \mathbb{N}$, for all $w_t \in \{j_t, k_t\}$, where ∞ in (62) means "undefined," and the limit "$\lim\limits_{s\to\infty}$" is taken under the usual topology on $\mathbb{R} \overset{\text{group}}{=} \mathscr{S}$.

Proof. First of all, observe that, for any $s \in \mathbb{R}$,

$$S_s\left(1_t^n\right) = 1_{t+s}^n = 1, \quad S_s\left(i_t^n\right) = i_{t+s}^n = i^n,$$

and

$$S_s\left(j_t^n\right) = j_{t+s}^n, \quad S_s\left(k_t^n\right) = k_{t+s}^n,$$

for all $n \in \mathbb{N}$, in the direct summand $\mathbb{H}_{t+s}$ of $\mathscr{H}$, since $S_s \in \mathscr{S}$ is identified with a $\mathbb{R}$-algebra-isomorphism $S_{t,t+s}$, the $(t, t+s)$-shift from $\mathbb{H}_t$ onto $\mathbb{H}_{t+s}$. It shows that

$$\lim_{s\to\infty} S_s\left(1_t^n\right) = \lim_{s\to\infty} 1_{t+s}^n = \lim_{s\to\infty} 1 = 1 = \left(\lim_{s\to\infty} S_s\left(1_t\right)\right)^n,$$

$$\lim_{s\to\infty} S_s\left(i_t^n\right) = \lim_{s\to\infty} i_{t+s}^n = \lim_{s\to\infty} i^n = i^n = \left(\lim_{s\to\infty} S\left(i_t\right)\right)^n,$$

where the second and the last equalities hold since $\{\mathbb{H}_{t+s}\}_{s \in \mathbb{R}}$ are $\mathbb{R}$-Banach algebras, and

$$\lim_{s\to\infty} S_s\left(j_t^n\right) = \lim_{s\to\infty} j_{t+s}^n = \left(\lim_{s\to\infty} j_{t+s}\right)^n = \left(\lim_{s\to\infty} S_s\left(j_t\right)\right)^n,$$

and

$$\lim_{s\to\infty} S_s\left(k_t^n\right) = \lim_{s\to\infty} k_{t+s}^n = \left(\lim_{s\to\infty} k_{t+s}\right)^n = \left(\lim_{s\to\infty} S_s\left(k_t\right)\right)^n,$$

i.e.,

$$\left(\lim_{s\to\infty} S_s\left(1_t\right)\right)^n = 1, \quad \left(\lim_{s\to\infty} S\left(i_t\right)\right)^n = i^n, \tag{63}$$

and

$$\left(\lim_{s\to\infty} S_s\left(j_t\right)\right)^n = \lim_{s\to\infty} j_{t+s}^n, \quad \left(\lim_{s\to\infty} S_s\left(k_t\right)\right)^n = \lim_{s\to\infty} k_{t+s}^n.$$

for all $n \in \mathbb{N}$. So,

$$\varphi\left(\left(\lim_{s\to\infty} S_s\left(1_t\right)\right)^n\right) = \tau_{t+s}\left(1\right) = 1,$$

and

$$\varphi\left(\left(\lim_{s\to\infty} S\left(i_t\right)\right)^n\right) = \tau_{t+s}\left(i^n\right) = \begin{cases} 0 & \text{if } n \in 2\mathbb{N}-1 \\ -1 & \text{if } n \in 2\mathbb{N} \setminus 4\mathbb{N} \\ 1 & \text{if } n \in 4\mathbb{N}, \end{cases}$$

by (59), for all $n \in \mathbb{N}$. Thus, the analytic data (60) and (61) hold.

Also, we have

$$\varphi\left(\left(\lim_{s\to\infty} S_s\left(j_t\right)\right)^n\right) = \lim_{s\to\infty} \tau_{t+s}\left(j_{t+s}^n\right), \tag{64}$$

and

$$\varphi\left(\left(\lim_{s\to\infty} S_s\left(k_t\right)\right)^n\right) = \lim_{s\to\infty} \tau_{t+s}\left(k_{t+s}^n\right),$$

by (63). i.e., if $w_t \in \{j_t, k_t\}$, then

$$\varphi\left(\left(\lim_{s\to\infty} S_s\left(w_t\right)\right)^n\right) = \lim_{s\to\infty} \tau_{t+s}\left(w_{t+s}^n\right),$$

by (64), since $(\mathscr{S}, \cdot) \overset{\text{group}}{=} (\mathbb{R}, +)$, and $\mathbb{R}$ is complete under its usual topology, and $\{\tau_t\}_{t\in\mathbb{R}}$ are bounded on $\{\mathbb{H}_t\}_{t\in\mathbb{R}}$. So,

$$\varphi\left(\left(\lim_{s\to\infty} S_s\left(w_t\right)\right)^n\right) = \lim_{s\to\infty} \tau_{t+s}\left(w_{t+s}^n\right) = \begin{cases} \lim_{s\to\infty} 0 & \text{if } n \in 2\mathbb{N}-1 \\ \lim_{s\to\infty} (t+s)^{\frac{n}{2}} & \text{if } n \in 2\mathbb{N} \end{cases}$$

by (59)

$$= \begin{cases} 0 & \text{if } n \in 2\mathbb{N}-1 \\ \infty & \text{if } n \in 2\mathbb{N}, \end{cases}$$

because if $s \to \infty$, then $t + s \to \infty$ in $\mathbb{R}$, for all arbitrarily fixed $t \in \mathbb{R}$. Therefore, the formula (62) holds. □

The above theorem not only provides the asymptotic analytic data (60), (61) and (62) on $\mathscr{H}$, but also lets us verify that if the scale t is suitably big in the sense that $t \to \infty$ in $\mathbb{R}$, then the analytic data on the t-scaled hypercomplexes $\mathbb{H}_t$ under the $\mathbb{R}$-trace τ_t becomes vague, especially, by (62). i.e., if t is suitably big in $\mathbb{R}$, then the analytic data $(\tau(h^n))_{n=1}^{\infty}$ of $h \in \mathbb{H}_t$ are mostly undefined to be ∞ by (62). Indeed, if

$$T = \underset{l=1}{\overset{N}{\oplus}} h_{t_l} \in \mathscr{H}, \text{ with } h_{t_l} \in \mathbb{H}_{t_l},$$

and if there exists at least one $t_{l_0} \in \{t_1, ..., t_N\}$, such that

$$h_{t_{l_0}} = x + yi_{t_{l_0}} + uj_{t_{l_0}} + vk_{t_{l_0}} \in \mathbb{H}_{t_{l_0}} \subset \mathscr{H},$$

with

$$\text{either } u \neq 0, \text{ or } v \neq 0, \text{ in } \mathbb{R},$$

then

$$\varphi\left(\left(\lim_{s\to\infty} S_s(T^n)\right)\right) = \sum_{l=1}^{N}\left(\lim_{s\to\infty} \tau_{\mathrm{t_l+s}}(h_{t_l+s}^n)\right) \to \infty,$$

by (62).

Corollary 2. Let $T = \underset{l=1}{\overset{N}{\oplus}} h_{t_l} \in \mathscr{H}$ with $h_{t_l} \in \mathbb{H}_{t_l}^{\times}$. Then

$$\left|\varphi\left(\left(\lim_{s\to\infty} S_s(T)\right)^n\right)\right| < \infty, \tag{65}$$

if and only if

$$h_{t_l} \in \mathrm{span}_{\mathbb{R}}\{1, i_{t_l} = i\} \subset \mathbb{H}_{t_l}, \ \forall l = 1, ..., N.$$

Proof. The boundedness characterization (65) holds true by (60), (61) and (62). □

The above corollary again illustrates that the φ-depending asymptotic analytic data becomes vague on the hypercomplex $\mathbb{R}$-algebra $\mathscr{H}$, in general.

Theorem 21. *Let $\{1_t = 1, i_t = i, j_t, k_t\}$ be the $\mathbb{R}$-basis of the t-scaled hypercomplexes $\mathbb{H}_t$, as a direct summand of the hypercomplex $\mathbb{R}$-algebra $\mathscr{H}$, for $t \in \mathbb{R}$. Then*

$$\left(\varphi\left(\left(\lim_{s\to-\infty} S_s(1_t)\right)^n\right)\right)_{n=1}^{\infty} = (\underline{1,1,1,1},1,1,...) \tag{66}$$

$$\left(\varphi\left(\left(\lim_{s\to-\infty} S_s(i_t)\right)^n\right)\right)_{n=1}^{\infty} = (\underline{0,-1,0,1},0,-1,0,1,...) \tag{67}$$

and

$$\varphi\left(\left(\lim_{s\to-\infty} S_s(w_t)\right)^n\right) = \begin{cases} 0 & \text{if } n \in 2\mathbb{N}-1 \\ -\infty & \text{if } n \in 2\mathbb{N} \setminus 4\mathbb{N} \\ \infty & \text{if } n \in 4\mathbb{N}, \end{cases} \tag{68}$$

for all $n \in \mathbb{N}$, for all $w_t \in \{j_t, k_t\}$.

Proof. Similar to the proof Theorem 38, one can get that

$$\left(\lim_{s\to-\infty} S_s\left(1_t\right)\right)^n = 1, \quad \left(\lim_{s\to-\infty} S\left(i_t\right)\right)^n = i^n,$$

and

$$\left(\lim_{s\to-\infty} S_s\left(j_t\right)\right)^n = \lim_{s\to\infty} j_{t+s}^n, \quad \left(\lim_{s\to-\infty} S_s\left(k_t\right)\right)^n = \lim_{s\to\infty} k_{t+s}^n.$$

So, we have

$$\varphi\left(\left(\lim_{s\to-\infty} S_s\left(1_t\right)\right)^n\right) = \mathrm{Re}\left(1\right) = 1,$$

and

$$\varphi\left(\left(\lim_{s\to-\infty} S\left(i_t\right)\right)^n\right) = \mathrm{Re}\left(i^n\right) = \begin{cases} 0 & \text{if } n \in 2\mathbb{N}-1 \\ -1 & \text{if } n \in 2\mathbb{N}\setminus 4\mathbb{N} \\ 1 & \text{if } n \in 4\mathbb{N}, \end{cases}$$

for all $n \in \mathbb{N}$. Thus, the analytic data (66) and (67) hold. Also, if $w_t \in \{j_t, k_t\}$, then

$$\varphi\left(\left(\lim_{s\to-\infty} S_s\left(w_t\right)\right)^n\right) = \mathrm{Re}\left(\lim_{s\to\infty} w_{t+s}^n\right),$$

where $w_{t+s} \in \{j_{t+s}, k_{t+s}\}$, respectively, for all $s \in \mathbb{R}$. Observe that

$$\varphi\left(\left(\lim_{s\to-\infty} S_s\left(w_t\right)\right)^n\right) = \mathrm{Re}\left(\lim_{s\to\infty} w_{t+s}^n\right) = \lim_{s\to\infty} \mathrm{Re}\left(w_{t+s}^n\right) = \begin{cases} \lim\limits_{s\to-\infty} 0 & \text{if } n \in 2\mathbb{N}-1 \\ \lim\limits_{s\to-\infty} \left(\mathrm{sgn}\left(t+s\right)\left|t+s\right|^{\frac{n}{2}}\right) & \text{if } n \in 2\mathbb{N}\setminus 4\mathbb{N} \\ \lim\limits_{s\to-\infty} \left|t+s\right|^{\frac{n}{2}} & \text{if } n \in 4\mathbb{N}, \end{cases}$$

by (59), where

$$\mathrm{sgn}\left(r\right) = \begin{cases} 1 & \text{if } r \geq 0 \\ -1 & \text{if } r < 0, \end{cases}$$

for all $r \in \mathbb{R}$, and hence, it goes to

$$= \begin{cases} 0 & \text{if } n \in 2\mathbb{N}-1 \\ -\infty & \text{if } n \in 2\mathbb{N}\setminus 4\mathbb{N} \\ \infty & \text{if } n \in 4\mathbb{N}, \end{cases}$$

for all $n \in \mathbb{N}$, because

$$\mathrm{sgn}\left(t+s\right) = -1, \quad \text{as} \quad s \to -\infty.$$

It shows that the formula (68) holds, too. □

This theorem not only gives the asymptotic analytic data (66), (67) and (68) on $\mathscr{H}$, but also makes us verify that if $|t|$ is suitably big, especially, $t \to -\infty$ in $\mathbb{R}$, then the analytic data on $\mathbb{H}_t$ up to the $\mathbb{R}$-trace τ_t becomes vague, in particular, by (68), implying that most of the analytic data on the hypercomplex $\mathbb{R}$-algebra $\mathscr{H}$ up to the $\mathbb{R}$-linear functional φ are undetermined, under the action of $\mathscr{S}$.

5.3. The Hypercomplex $[-1,1]$-Algebra $\mathscr{H}[-1,1]$

In Section 5.2, we considered the asymptotic analytic data on the hypercomplex $\mathbb{R}$-algebra $\mathscr{H} = \underset{t\in\mathbb{R}}{\oplus^{a}}\mathbb{H}_t$ up to the $\mathbb{R}$-trace $\varphi = \underset{t\in\mathbb{R}}{\oplus}\tau_t$, under the dynamical action of $(\mathscr{S},\cdot) \overset{\text{group}}{=} (\mathbb{R},+)$. The main results there showed that most asymptotic analytic data of the non-zero elements $T \in \mathscr{H}$ are undefined up to φ, especially, by (62) and (68). Motivated by these asymptotic information, we construct a sub-structure $\mathscr{H}[-1,1]$ of $\mathscr{H}$, where $[-1,1] = \{r \in \mathbb{R} : -1 \leq r \leq 1\}$ be the closed interval of $\mathbb{R}$. Define $\mathscr{H}[-1,1]$ by a $\mathbb{R}$-algebra,

$$\mathscr{H}[-1,1] \overset{\text{def}}{=} \underset{t\in[-1,1]}{\oplus^{a}}\mathbb{H}_t, \text{ in } \mathscr{H}. \tag{69}$$

By (69), this $\mathbb{R}$-algebra $\mathscr{H}[-1,1]$ is a subalgebra of $\mathscr{H}$. Of course, similar to (69), one can define the $\mathbb{R}$-subalgebras,

$$\mathscr{H}[t_1,t_2] = \underset{t\in[t_1,t_2]}{\oplus^{a}}\mathbb{H}_t \text{ of } \mathscr{H},$$

for any $t_1 \leq t_2$ in $\mathbb{R}$, axiomatizing $\mathscr{H}_{[t,t]} = \mathbb{H}_t$, for all $t \in \mathbb{R}$. There are no typical reasons why we take the closed interval $[-1,1]$ in (69), instead of taking arbitrary closed intervals of $\mathbb{R}$. However, one may / can realize that this direct product algebra $\mathscr{H}[-1,1]$ is constructed by the pure-algebraic direct product "from the quaternions $\mathbb{H}_{-1}$ to the split-quaternions $\mathbb{H}_1$," in $\mathscr{H}$, induced by negative scales, the 0-scale, and positive scales, all together. Moreover, one can avoid the vague asymptotic analytic data on $\mathscr{H}$ up to φ in $\mathscr{H}[-1,1]$. See the following result.

Corollary 3. For all $t \in [-1,1]$, if $1_t = 1$ and $i_t = i$ in $\mathbb{H}_t$, then

$$\begin{aligned}
(\varphi(1_t^n))_{n=1}^{\infty} &= \left(\underline{1,1,1,1},1,1,1,1,\ldots\right), \\
(\varphi(i_t^n))_{n=1}^{\infty} &= \left(\underline{0,-1,0,1},0,-1,0,1,\ldots\right), \\
\varphi(j_t^n) = \varphi(k_t^n) &= \begin{cases} 0 & \text{if } n \in 2\mathbb{N}-1 \\ t^{\frac{n}{2}} & \text{if } n \in 2\mathbb{N}, \end{cases}
\end{aligned} \tag{70}$$

for all $n \in \mathbb{N}$, where

$$-1 \leq t^m \leq 1, \quad \forall t \in [-1,1], \ \forall m \in \mathbb{N}. \tag{71}$$

Proof. The proofs of (70) are done by (59). The boundedness condition (71) for the formulas (70) is trivial since $t \in [-1,1]$. □

In fact, the condition (71) on (70) allows us to avoid the undefined asymptotic analytic data up to φ.

Definition 11. The subalgebra $\mathscr{H}[-1,1]$ of (69) is called the hypercomplex $[-1,1]$-algebra (over $\mathbb{R}$ in the hypercomplex $\mathbb{R}$-algebra $\mathscr{H}$).

6. On the Hypercomplex $[-1,1]$-Algebra $\mathscr{H}[-1,1]$

Let $\mathscr{H}[-1,1] = \underset{t\in[-1,1]}{\oplus^{a}}\mathbb{H}_t$ be the hypercomplex $[-1,1]$-algebra (69) embedded in the hypercomplex $\mathbb{R}$-algebra $\mathscr{H}$. Note that, on $\mathscr{H}[-1,1]$, the analytic data (70) holds under the boundedness condition (71), up to the (restriction of the) $\mathbb{R}$-linear functional φ. Let

$$h_t = x_1 + x_{i_t} i_t + x_{j_t} j_t + x_{k_t} k_t \in \mathbb{H}_t \text{ in } \mathscr{H}\left[-1,1\right], \tag{72}$$

for $t \in [-1,1]$ in $\mathbb{R}$, where $x_1, x_{i_t}, x_{j_t}, x_{k_t} \in \mathbb{R}$. Let

$$\zeta_t \overset{\text{def}}{=} \begin{cases} \frac{j_t}{\sqrt{|t|}} & \text{if } t \neq 0 \\ j_0 & \text{if } t = 0, \end{cases} \tag{73}$$

and

$$\kappa_t \overset{\text{def}}{=} \begin{cases} \frac{k_t}{\sqrt{|t|}} & \text{if } t \neq 0 \\ k_0 & \text{if } t = 0, \end{cases}$$

in $\mathbb{H}_t$. Then the elements $i_t = i$, ζ_t and κ_t satisfy that

$$i_t^2 = -1, \quad \zeta_t^2 = s_0\left(t\right) = \kappa_t^2, \tag{74}$$

and

$$\begin{array}{ccc} & i_t & \\ {}^{1}\swarrow & & \nwarrow^{-s_0(t)} \\ \zeta_t & \underset{1}{\longrightarrow} & \kappa_t \end{array} \quad \text{and} \quad \begin{array}{ccc} & i_t & \\ {}^{s_0(t)}\nearrow & & \searrow^{-1} \\ \zeta_t & \underset{-1}{\longleftarrow} & \kappa_t \end{array},$$

where

$$s_0\left(t\right) = \begin{cases} 1 & \text{if } t > 0 \\ -1 & \text{if } t < 0 \\ 0 & \text{if } t = 0, \end{cases}$$

for all $t \in \mathbb{R}$, where the first diagram of (74) means that

$$i_t \zeta_t = \kappa_t, \quad \zeta_t \kappa_t = -s_0\left(t\right) i_t, \quad \kappa_t i_t = \zeta_t,$$

and the second diagram of (74) means that

$$i_t \kappa_t = -\zeta_t, \quad \kappa_t \zeta_t = s_0\left(t\right) i_t, \quad \zeta_t i_t = -\kappa_t,$$

by (19). In particular, the first line of (74) holds because

$$\zeta_t^2 = \begin{cases} \left(\frac{j_t}{\sqrt{|t|}}\right)^2 = \frac{t}{|t|} \in \{\pm 1\} & \text{if } t \neq 0 \\ j_0^2 = 0 & \text{if } t = 0, \end{cases}$$

and

$$\kappa_t^2 = \begin{cases} \left(\frac{k_t}{\sqrt{|t|}}\right)^2 = \frac{t}{|t|} \in \{\pm 1\} & \text{if } t \neq 0 \\ k_0^2 = 0 & \text{if } t = 0, \end{cases}$$

for all $t \in \mathbb{R}$. If ζ_t and κ_t are in the sense of (73) in $\mathbb{H}_t$, then the element $h_t \in \mathbb{H}_t$ of (72) can be re-expressed to be

$$h_t = x_1 + x_{i_t} i_t + \widehat{x_{j_t}} \zeta_t + \widehat{x_{k_t}} \kappa_t, \tag{75}$$

with

$$\widehat{x_{j_t}} = \begin{cases} x_{j_t}\sqrt{|t|} & \text{if } t \neq 0 \\ x_{j_0} & \text{if } t = 0, \end{cases}$$

and

$$\widehat{x_{k_t}} = \begin{cases} x_{k_t}\sqrt{|t|} & \text{if } t \neq 0 \\ x_{k_0} & \text{if } t = 0. \end{cases}$$

Note that the function "$x \in \mathbb{R} \longmapsto x\sqrt{|t|} \in \mathbb{R}$" is bijective on $\mathbb{R}$, whenever $t \neq 0$. And hence, without loss of generality, the element $h_t \in \mathbb{H}_t$ of (72) is always expressed to be (75), where $\{i_t, \zeta_t, \kappa_t\}$ satisfy the relation (74). i.e.,

$$\mathbb{H}_t \overset{\text{def}}{=} \text{span}_{\mathbb{R}} \{1, i_t, j_t, k_t\} \overset{\text{iso}}{=} \text{span}_{\mathbb{R}} \{1, i_t, \zeta_t, \kappa_t\}, \tag{76}$$

where ζ_t and κ_t are in the sense of (73), for all $t \in [-1, 1]$ (in fact, for all $t \in \mathbb{R}$). Then, by (70), we obtain the following result.

Theorem 22. *Let* $\{1, i_t = i, \zeta_t, \kappa_t\} \subset \mathbb{H}_t$*, where* ζ_t and κ_t *are in the sense of* (73)*, in the hypercomplex* $[-1, 1]$*-algebra* $\mathscr{H}[-1, 1]$*. Then*

$$\begin{aligned} (\varphi(1^n))_{n=1}^{\infty} &= \left(\underline{1, 1, 1, 1}, 1, 1, 1, 1, ...\right), \\ (\varphi(i_t^n))_{n=1}^{\infty} &= \left(\underline{0, -1, 0, 1}, 0, -1, 0, 1, ...\right), \\ (\varphi(\zeta_t^n))_{n=1}^{\infty} &= (\varphi(\kappa_t^n))_{n=1}^{\infty} = \left(\underline{0, s_0(t), 0, s_0(t)^2}, 0, s_0(t), ...\right). \end{aligned} \tag{77}$$

Proof. The first two analytic sequences of (77) are obtained directly by (70). Meanwhile, if $w \in \{\zeta_t, \kappa_t\}$, then

$$w^n = \begin{cases} s_0(t)^{\frac{n-1}{2}} w & \text{if } n \in 2\mathbb{N} - 1 \\ s_0(t)^{\frac{n}{2}} & \text{if } n \in 2\mathbb{N}, \end{cases}$$

in $\mathbb{H}_t \subset \mathscr{H}[-1, 1]$ by (74), for all $n \in \mathbb{N}$, implying that

$$\varphi(w^n) = \tau_t(w^n) = \text{Re}(w^n) = \begin{cases} 0 & \text{if } n \in 2\mathbb{N} - 1 \\ s_0(t)^{\frac{n}{2}} & \text{if } n \in 2\mathbb{N}, \end{cases}$$

for all $n \in \mathbb{N}$. Remark that, since $s_0(t) \in \{-1, 0, 1\}$ for a fixed $t \in \mathbb{N}$,

$$s_0(t)^{2k-1} = s_0(t), \text{ and } s_0(t)^{2k} = |s_0(t)| = s_0(t)^2,$$

in $\{-1, 0, 1\}$, for all $k \in \mathbb{N}$. Thus, one has that

$$\varphi(w^n) = \begin{cases} 0 & \text{if } n \in 2\mathbb{N} - 1 \\ s_0(t) & \text{if } n \in 2\mathbb{N} \setminus 4\mathbb{N} \\ |s_0(t)| = s_0(t)^2 & \text{if } n \in 4\mathbb{N}, \end{cases}$$

for all $n \in \mathbb{N}$. Therefore, the last analytic sequence of (77) holds. □

The last analytic sequence of (77) can be refined as follows: (i) if $t > 0$, then

$$\left(\underline{0, 1, 0, 1}, 0, 1, 0, 1, ...\right);$$

and (ii) if $t < 0$, then

$$\left(\underline{0, -1, 0, 1}, 0, -1, 0, 1, ...\right);$$

and (iii) if $t = 0$, then

$$\left(\underline{0, 0, 0, 0}, 0, 0, 0, 0, ...\right).$$

Now, let $t \in [-1, 1]$, and

$$h_t = x_1 + x_{i_t} i_t + x_{\zeta_t} \zeta_t + x_{\kappa_t} \kappa_t \in \mathbb{H}_t \tag{78}$$

under the relation (76). If we let

$$\mathcal{B}_t \overset{\text{denote}}{=} \{1, i_t, \zeta_t, \kappa_t\} \subset \mathbb{H}_t,$$

then the element $h_t \in \mathbb{H}_t$ of (78) satisfies that

$$h_t^n = \sum_{w \in \mathcal{B}_t} \left(\sum_{(w_1,...,w_n) \in \mathcal{B}_t^n,\ \prod_{l=1}^n w_l = w} \left(\prod_{l=1}^n x_{w_l} \right) - \sum_{(w_1,...,w_n) \in \mathcal{B}_t^n,\ \prod_{l=1}^n w_l = -w} \left(\prod_{l=1}^n x_{w_l} \right) \right) w, \tag{79}$$

in $\mathbb{H}_t \subset \mathscr{H}[-1, 1]$, having their real part,

$$\mathrm{Re}\left(h_t^n\right) = \sum_{(w_1,...,w_n) \in \mathcal{B}_t^n,\ \prod_{l=1}^n w_l = 1} \left(\prod_{l=1}^n x_{w_l} \right) - \sum_{(w_1,...,w_n) \in \mathcal{B}_t^n,\ \prod_{l=1}^n w_l = -1} \left(\prod_{l=1}^n x_{w_l} \right) \tag{80}$$

which is identified with $\tau_t\left(h_t^n\right) = \varphi\left(h_t^n\right)$, for all $n \in \mathbb{N}$.

Lemma 8. Let $h_t \in \mathbb{H}_t$ be in the sense of (72) in $\mathscr{H}[-1, 1]$, for $t \in [-1, 1]$ in $\mathbb{R}$. Then

$$\varphi\left(h_t^n\right) = \sum_{(w_1,...,w_n) \in \mathcal{B}_t^n,\ \prod_{l=1}^n w_l = 1} \left(\prod_{l=1}^n x_{w_l} \right) - \sum_{(w_1,...,w_n) \in \mathcal{B}_t^n,\ \prod_{l=1}^n w_l = -1} \left(\prod_{l=1}^n x_{w_l} \right), \tag{81}$$

for all $n \in \mathbb{N}$, where $\mathcal{B}_t = \{1, i_t, \zeta_t, \kappa_t\}$ is in the sense of (79).

Proof. The analytic data (81) is obtained by (80) in $\mathscr{H}[-1, 1]$ up to φ, since

$$\varphi\left(h_t^n\right) = \tau_t\left(h_t^n\right) = \mathrm{Re}\left(h_t^n\right), \quad \forall n \in \mathbb{N},$$

for all $h_t \in \mathbb{H}_t$ in $\mathscr{H}[-1, 1]$, for $t \in [-1, 1]$. □

By the above lemma, we obtain the following general result.

Theorem 23. *Let* $T = \overset{N}{\underset{l=1}{\oplus}} h_{t_l} \in \mathscr{H}[-1, 1]$, *for* $t_1, ..., t_N \in [-1, 1]$ *and* $N \in \mathbb{N}$, *where*

$$h_{t_l} = x_1^{(t_l)} + x_{i_{t_l}}^{(t_l)} i_{t_l} + x_{j_{t_l}}^{(t_l)} \zeta_{t_l} + x_{k_{t_l}}^{(t_l)} \kappa_{t_l} \in \mathbb{H}_{t_l}, \tag{82}$$

with $x_{w_{t_l}}^{(t_l)} \in \mathbb{R}$, *for all* $l = 1, ..., N$. *Then*

$$\varphi\left(T^{n}\right)=\sum_{l=1}^{N}\left(\sum_{(w_1,...,w_n)\in\mathcal{B}_{t_l}^{n},\ \prod_{l=1}^{n}w_l=1}\left(\prod_{l=1}^{n}x_{w_l}^{(t_l)}\right)-\sum_{(w_1,...,w_n)\in\mathcal{B}_{t_l}^{n},\ \prod_{l=1}^{n}w_l=-1}\left(\prod_{l=1}^{n}x_{w_l}^{(t_l)}\right)\right), \tag{83}$$

for all $n\in\mathbb{N}$, where $\mathcal{B}_{t_l}=\{1,i_{t_l},\zeta_{t_l},\kappa_{t_l}\}\subset\mathbb{H}_{t_l}$ are in the sense of (79) for all $l=1,...,N$.

Proof. Under hypothesis, one has

$$\varphi\left(T^{n}\right)=\sum_{l=1}^{N}\tau_{t_l}\left(h_{t_l}^{n}\right)=\sum_{l=1}^{N}\mathrm{Re}\left(h_{t_l}^{n}\right),\ \forall n\in\mathbb{N},$$

since

$$T^{n}=\left(\underset{l=1}{\overset{N}{\oplus}}h_{t_l}\right)^{n}=\underset{l=1}{\overset{N}{\oplus}}h_{t_l}^{n}\ \text{ in }\ \mathscr{H},\ \ \forall n\in\mathbb{N}.$$

Thus, the analytic data (83) holds by (81) and (82). □

Now, let

$$\mathfrak{S}[-1,1]=\{\sigma:[-1,1]\to[-1,1]\,|\,\sigma\text{ is bijective}\}.$$

Then one can define a morphism $\Phi_\sigma:\mathscr{H}[-1,1]\to\mathscr{H}[-1,1]$ by

$$\Phi_\sigma\left(\underset{t\in[-1,1]}{\oplus}h_t\right)\overset{\text{def}}{=}\underset{t\in[-1,1]}{\oplus}h_{\sigma(t)},\quad\forall\underset{t\in[-1,1]}{\oplus}h_t\in\mathscr{H}[-1,1], \tag{84}$$

where if $h_t=x+yi_t+u\zeta_t+v\kappa_t\in\mathbb{H}_t$, under the relation (76), then

$$h_{\sigma(t)}\overset{\text{denote}}{=}\Phi_\sigma\left(h_t\right)=x+yi_{\sigma(t)}+u\zeta_{\sigma(t)}+v\kappa_{\sigma(t)},$$

for $x,y,u,v\in\mathbb{R}$, for all $\sigma\in\mathfrak{S}[-1,1]$. Then it is not difficult to check that

$$\Phi_\sigma\left(r_1T_1+r_2T_2\right)=r_1\Phi_\sigma\left(T_1\right)+r_2\Phi_\sigma\left(T_2\right), \tag{85}$$

and

$$\Phi_\sigma\left(T_1T_2\right)=\Phi_\sigma\left(T_1\right)\Phi_\sigma\left(T_2\right), \tag{86}$$

by (84), for all $r_1,r_2\in\mathbb{R}$ and $T_1,T_2\in\mathscr{H}[-1,1]$.

Suppose A is an arbitrary $\mathbb{R}$-algebra, and let

$$Aut_{\mathbb{R}}\left(A\right)=\{\Psi:A\to A\,|\Psi\text{ is a }\mathbb{R}\text{-algebra-isomorphism}\}$$

be the automorphism group on A, consisting of all (pure-algebraic) $\mathbb{R}$-algebra-isomorphisms on A, equipped with the isomorphism multiplication $(\cdot)$ (or, the composition).

Proposition 24. *The family $\mathscr{S}[-1,1]\overset{denote}{=}\{\Phi_\sigma:\sigma\in\mathfrak{S}[-1,1]\}$ forms a subgroup $(\mathscr{S}[-1,1],\cdot)$ of the automorphism group $Aut_{\mathbb{R}}(\mathscr{H}[-1,1])$, where $\Phi_\sigma\in\mathscr{S}[-1,1]$ are in the sense of (84). i.e.,*

$$\mathscr{S}[-1,1]\overset{group}{\subseteq}Aut\left(\mathscr{H}[-1,1]\right), \tag{87}$$

where "$\overset{group}{\subseteq}$" means "being a subgroup of."

Proof. By (85) and (86), each element Φ_σ of $\mathscr{S}[-1,1]$ is a well-defined bijective multiplicative $\mathbb{R}$-linear transformation on $\mathscr{H}[-1,1]$, equivalently, it is a (pure-algebraic) $\mathbb{R}$-algebra-isomorphism (or, a $\mathbb{R}$-automorphism) on $\mathscr{H}[-1,1]$. So,

$$\mathscr{S}[-1,1] \subseteq Aut\left(\mathscr{H}[-1,1]\right), \text{ set-theoretically.}$$

Now, let $\Phi_{\sigma_1}, \Phi_{\sigma_2} \in \mathscr{S}[-1,1]$. Then, for any $T = \underset{t\in[-1,1]}{\oplus} h_t \in \mathscr{H}[-1,1]$, we have

$$\left(\Phi_{\sigma_1}\Phi_{\sigma_2}\right)(T) = \Phi_{\sigma_1}\left(\underset{t\in[-1,1]}{\oplus} h_{\sigma_2(t)}\right) = \underset{t\in[-1,1]}{\oplus} h_{(\sigma_1\circ\sigma_2)(t)},$$

in $\mathscr{H}[-1,1]$, implying that

$$\Phi_{\sigma_1}\Phi_{\sigma_2} = \Phi_{\sigma_1\circ\sigma_2} \text{ on } \mathscr{H}[-1,1],$$

for all $\sigma_1, \sigma_2 \in \mathfrak{S}[-1,1]$, where $\sigma_1 \circ \sigma_2$ is the composition of the bijections σ_1 and σ_2 in $\mathfrak{S}[-1,1]$. Remark that, for any $\sigma \in \mathfrak{S}[-1,1]$, we have

$$\Phi_\sigma^{-1} = \Phi_{\sigma^{-1}} \in \mathscr{S}[-1,1].$$

So,

$$\Phi_{\sigma_1}\Phi_{\sigma_2}^{-1} = \Phi_{\sigma_1}\Phi_{\sigma_2^{-1}} = \Phi_{\sigma_1\circ\sigma_2^{-1}} \in \mathscr{S}[-1,1],$$

in $Aut_{\mathbb{R}}\left(\mathscr{H}[-1,1]\right)$. Therefore,

$$\mathscr{S}[-1,1] \overset{\text{group}}{\subseteq} Aut_{\mathbb{R}}\left(\mathscr{H}[-1,1]\right),$$

proving the relation (87). □

From below, we understand $\mathscr{S}[-1,1] = \{\Phi_\sigma : \sigma \in \mathfrak{S}[-1,1]\}$ as a subgroup of the automorphism group $Aut_{\mathbb{R}}\left(\mathscr{H}[-1,1]\right)$ by (87). Now, let

$$h_t = x + yi_t + u\zeta_t + v\kappa_t \in \mathbb{H}_t \text{ in } \mathscr{H}[-1,1],$$

where $x, y, u, v \in \mathbb{R}$, and ζ_t and κ_t are in the sense of (73). Then, for any $\Phi_\sigma \in \mathscr{S}[-1,1]$,

$$h_{\sigma(t)} \overset{\text{denote}}{=} \Phi_\sigma(h_t) = x + yi_{\sigma(t)} + u\zeta_{\sigma(t)} + v\kappa_t \in \mathbb{H}_{\sigma(t)},$$

in $\mathscr{H}[-1,1]$, satisfying

$$\begin{aligned} \left(\varphi\left(i_{\sigma(t)}^n\right)\right)_{n=1}^\infty &= \left(\varphi\left(i^n\right)\right)_{n=1}^\infty = \left(\underline{0,-1,0,1},0,-1,0,1,...\right), \\ \left(\varphi\left(\zeta_{\sigma(t)}^n\right)\right)_{n=1}^\infty &= \left(\varphi\left(\kappa_{\sigma(t)}^n\right)\right)_{n=1}^\infty = \left(\underline{0, s_0(\sigma(t)), 0, s_0(\sigma(t))^2}, ...\right), \end{aligned}$$

for all $n \in \mathbb{N}$, by (77).

Proposition 25. *Assume that $\sigma \in \mathfrak{S}[-1,1]$ has its fixed point at $t \in [-1,1]$ in the sense that: $\sigma(t) = t$ in $[-1,1]$. Then, for any $h \in \mathbb{H}_t \subset \mathscr{H}[-1,1]$, we have*

$$\varphi\left(h^n\right) = \varphi\left(\Phi_\sigma(h)^n\right) \text{ in } \mathscr{H}[-1,1], \ \forall n \in \mathbb{N}. \tag{88}$$

Proof. Assume that $t \in [-1,1]$ is the fixed point of $\sigma \in \mathfrak{S}[-1,1]$, i.e., $\sigma(t) = t$ in $[-1,1]$. Then,

$$\sigma\left(\mathbb{H}_t\right) = \mathbb{H}_{\sigma(t)} = \mathbb{H}_t, \text{ in } \mathscr{H}[-1,1],$$

satisfying

$$\Phi_\sigma(h) = h \in \mathbb{H}_t \text{ in } \mathscr{H}[-1,1], \ \forall h \in \mathbb{H}_t.$$

Therefore, the analytic data (88) holds. □

Now, we consider how the analytic data (83) of $\mathscr{H}[-1,1]$ are affected by the action of $\mathscr{S}[-1,1]$.

Theorem 26. *Let $h_t = x_1 + x_{i_t} i_t + x_{\zeta_t}\zeta_t + x_{\kappa_t}\kappa_t \in \mathbb{H}_t$ be an element of $\mathscr{H}[-1,1]$, with $x_1, x_{i_t}, x_{\zeta_t}, x_{\kappa_t} \in \mathbb{R}$, and $\Phi_\sigma \in \mathscr{S}[-1,1]$, for $\sigma \in \mathfrak{S}[-1,1]$. Then*

$$\varphi\left(\Phi_\sigma\left(h_t\right)^n\right) = \sum_{(w_1,\dots,w_n)\in\mathcal{B}^n_{\sigma(t)},\ \prod_{l=1}^{n} w_l = 1} \left(\prod_{l=1}^{n} x_{w_l}\right) - \sum_{(w_1,\dots,w_n)\in\mathcal{B}^n_{\sigma(t)},\ \prod_{l=1}^{n} w_l = -1} \left(\prod_{l=1}^{n} x_{w_l}\right), \tag{89}$$

for all $n \in \mathbb{N}$, where $\mathcal{B}_{\sigma(t)} = \left\{1, i_{\sigma(t)} = i, \zeta_{\sigma(t)}, \kappa_{\sigma(t)}\right\}$ in $\mathbb{H}_{\sigma(t)}$.

Proof. The formula (89) holds by (83). □

By (89), we immediately obtain the following corollary.

Corollary 4. Let $T = \underset{l=1}{\overset{N}{\oplus}} h_{t_l} \in \mathscr{H}[-1,1]$, for $t_1, \dots, t_N \in [-1,1]$ and $N \in \mathbb{N}$, where

$$h_{t_l} = x_1^{(t_l)} + x_{i_{t_l}}^{(t_l)} i_{t_l} + x_{j_{t_l}}^{(t_l)} j_{t_l} + x_{k_{t_l}}^{(t_l)} k_{t_l} \in \mathbb{H}_{t_l}, \tag{90}$$

with $x_{w_{t_l}}^{(t_l)} \in \mathbb{R}$, for all $l = 1, \dots, N$. If $\Phi_\sigma \in \mathscr{S}[-1,1]$, then

$$\varphi\left(\Phi_\sigma\left(T\right)^n\right) = \sum_{l=1}^{N} \left(\sum_{(w_1,\dots,w_n)\in\mathcal{B}^n_{t},\ \prod_{l=1}^{n} w_l = 1} \left(\prod_{l=1}^{n} x_{w_l}\right) - \sum_{(w_1,\dots,w_n)\in\mathcal{B}^n_{t},\ \prod_{l=1}^{n} w_l = -1} \left(\prod_{l=1}^{n} x_{w_l}\right)\right), \tag{91}$$

for all $n \in \mathbb{N}$.

Proof. The analytic data (91) holds, by (90). □

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