Advances in Analysis and Applied Mathematics, **1**(1) (2024), 12–18.

https://doi.org/10.62298/advmath.5
ISSN Online: 3062-0686 / Open Access

ADVANCES IN ANALYSIS AND APPLIED MATHEMATICS

Check for updates

Research Paper

Subclasses of λ-Pseudo Starlike Functions With Respect to Symmetric Points Associated With Conic Region

Musthafa Ibrahim[1,*]

[1]College of Engineering, University of Buraimi, Al Buraimi, Sultanate of Oman
[*]To whom correspondence should be addressed: musthafa.i@uob.edu.om

Received on 1 March 2024; Revised on 15 April 2024; Accepted on 8 May 2024; Published on 30 June 2024

Abstract

The purpose of this paper is to introduce and study a new subclass $\rho\tau\kappa^{\lambda}_{s,\Sigma}(\alpha, P(z))$ of the class Σ of bi-univalent functions defined in the unit disk, called λ-bi-pseudo-starlike, with respect to symmetric points associated with conic region impacted by Janowski functions. Further we determine the Fekete-Szegö result for the function class.

Key Words: Analytic Functions, Bi-Univalent, Fekete-Szegö, Coefficient Inequalities, Starlike Functions and Convex Functions, Subordination
AMS 2020 Classification: 30C45, 30C50, 30C80, 11B65, 47B38

1. Introduction and Motivation

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

analytic in the open unit disk $\mathcal{U}$. Also we let $\mathcal{S}$ denote the class of all function in $\mathcal{A}$ which are univalent in $\mathcal{U}$. It is well known that every function $f \in S$ has a function f^{-1}, defined by

$$f^{-1}[f(z)] = z\,; (z \in \mathcal{U})$$

and

$$f[f^{-1}(w)] = w\,; \qquad (\,|w| < r_0(f)\,;\, r_0 f \geq \frac{1}{4}\,).$$

In fact, the inverse function f^{-1} is given by

$$f^{-1}(w) = w - a_2 w^2 + (2a_2 w^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots .$$

A function $f \in \mathcal{A}$ is said to be biunivalent in $\mathcal{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathcal{U}$. Let f and g be analytic in the open unit disk $\mathcal{U}$. The function f is subordinate to g written as $f \prec g$ in $\mathcal{U}$, if there exist a function w analytic in $\mathcal{U}$ with $w\,(0) = 0$ and $|w(z)| < 1$; $(z \in \mathcal{U})$ such that $f(z) = g(w(z)), (z \in \mathcal{U})$.

Using the concept of subordination for holomorphic functions, Ma and Minda [1] introduced the classes

$$S^{*}(\psi)=\left\{f\in\mathcal{A}:\ Re\left(\frac{z\,f'(z)}{f(z)}\right)\prec\psi(z)\right\}$$

and

$$C(\psi)=\left\{f\in\mathcal{A}:\ Re\left(1+\frac{z\,f''(z)}{f(z)}\right)\prec\psi(z)\right\}.$$

where $\psi\in\wp$ with $\psi\text{'}>0$ maps $\mathcal{U}$ onto a region starlike with respect to 1 and symmetric with respect to real axis. By choosing ψ to map unit disc on to some specific regions like parabolas, cardioid, lemniscate of Bernoulli, booth lemniscate in the right-half of the complex plane, various interesting subclasses of starlike and convex functions can be obtained.

For arbitrary fixed numbers $C,D,-1<C\leq 1,-1\leq D<C$, we denote $\wp(C,D)$ by the family of functions $p(z)=1+p_1z+p_2z^2+\cdots$ analytic in the unit disc and $p(z)\in\wp(C,D)$ if and only if

$$p(z)=\frac{1+Cw(z)}{1+Dw(z)},$$

where $w(z)$ is the Schwarz function. On observing that $w(z)=\frac{p(z)-1}{p(z)+1}$ for $p(z)\in\wp$, we have $S(z)\in\wp(C,D)$ if and only if for some $p(z)\in\wp$

$$S(z)=\frac{(1+C)p(z)+1-C}{(1+D)p(z)+1-D}$$

$$S^{*}(\psi)=\left\{f\in\mathcal{A}:\ Re\left(\frac{z\,f'(z)}{f(z)}\right)\prec\frac{1+Cz}{1+Dz},-1\leq D<C\leq 1\right\}$$

and

$$C(\psi)=\left\{f\in\mathcal{A}:\ Re\left(1+\frac{z\,f''(z)}{f(z)}\right)\prec\frac{1+Cz}{1+Dz},-1\leq D<C\leq 1,\right\}.$$

respectively.

Motivated by aforementioned works [2, 3, 4, 5, 6, 7], in this paper we defined the following new subclass $\rho\tau\kappa^{\lambda}_{s,\Sigma}(\alpha,P(z))$ named as λ-pseudo-starlike function of the class Σ of bi-univalent functions defined in the unit disk,with respect to symmetric points associated with conic region impacted by Janowski functions.

Definition 1. [8]

For $0\leq\alpha\leq 1;\lambda>0;\lambda\neq\frac{1}{3}$ a function $f\in\Sigma$ of the form (1) is said to be in the class $\rho\tau\kappa^{\lambda}_{s,\Sigma}(\alpha,P(z))$ if the following subordination hold:

$$\left(\frac{2z\left(f'(z)\right)^{\lambda}}{f(z)-f(-z)}\right)^{\alpha}\left(\frac{2\left(zf'(z)\right)^{\lambda}}{[f(z)-f(-z)]'}\right)^{1-\alpha}\prec\frac{(C+1)\psi(z)-(C-1)}{(D+1)\psi(z)-(D-1)}$$

Specializing the parameter $\lambda=1$ we have the following definitions, respectively:

Definition 2. For $0\leq\alpha\leq 1$ a function $f\in\Sigma$ of the form (1) is said to be in the class $\rho\tau\kappa^{1}_{s,\Sigma}(\alpha,P(z))\equiv\rho\tau\kappa_{s,\Sigma}(\alpha,P(z))$ if the following subordination hold:

$$\left(\frac{2zf'(z)}{f(z)-f(-z)}\right)^{\alpha}\left(\frac{2zf'(z)}{[f(z)-f(-z)]'}\right)^{1-\alpha}\prec\frac{(C+1)\psi(z)-(C-1)}{(D+1)\psi(z)-(D-1)}$$

Further by specializing the parameter $\alpha=1$ and $\alpha=0$ we state the following new classes

Definition 3. A function $f \in \Sigma$ of the form (1) is said to be in the class $\rho\tau\kappa^{1}_{s,\Sigma}(1, P(z)) \equiv \rho\tau\kappa_{s,\Sigma}(P(z))$ if the following subordination hold:

$$\frac{2zf'(z)}{f(z)-f(-z)} \prec \frac{(C+1)\psi(z)-(C-1)}{(D+1)\psi(z)-(D-1)}$$

2. Preliminaries

In this section we state the results that would be used to establish our main results which can be found in the standard text on univalent function theory.

Lemma 1. [9]If the function $f \in \mathcal{A}$ given by and g given by

$$g\,(w) = w + \sum_{n=2}^{\infty} b_k\, w^n$$

is inverse function, then the coefficients b_k, for $k \geq 2$, are given by

$$b_k = \frac{(-a)^{k+1}}{k!}\begin{pmatrix} ka_2 & 1 & 0 & \cdots & 0 \\ 2ka_2 & (K+1)a_2 & 2 & \cdots & 0 \\ 3ka_4 & (2K+1)a_3 & (K+2)a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ (k-1)ka_k & [k(k-2)+1]\,a_{k-1} & [k(k-3)+2]\,a_{k-2} & \cdots & (2k-2)a_2 \end{pmatrix} \tag{2}$$

Lemma 2. [10] If $p(z) = 1 + \sum_{n=1}^{\infty} P_k\, z^k \in \psi$, then $\mid p_k \mid \leq 2$ for $k \geq 1$ and the inequality is sharp for $p_\lambda(z) = \frac{1+\lambda z}{1-\lambda z}, \mid \lambda \mid \leq 1$.

Lemma 3. [1] If $p(z) = 1 + \sum_{n=1}^{\infty} P_k\, z^k \in \psi$, and v is complex number, then

$$\mid p_2 - vp_1^2 \mid \leq 2max[1; \mid 2v-1 \mid]$$

3. Main Result

Theorem 1. *Let $f \in \rho\tau\kappa^{\lambda}_{s,\Sigma}(\alpha, P(z))$, is given by (1) , then for the coefficients of $g = f^{-1}$ the following estimates hold:*

$$\mid b_2 \mid \leq \frac{\mid L_1 \mid (C-D)}{4 \mid \lambda \mid \mid \alpha - 2 \mid} \tag{3}$$

and

$$\mid b_3 \mid \leq \frac{\mid L_1 \mid (C-D)}{\mid 4(3\lambda-1)(3-2\lambda) \mid} max\,[1; \mid 2v-1 \mid] \tag{4}$$

with

$$v := \frac{1}{4}\left[L_1(D+1) + 2\left(1 - \frac{L_2}{L_1}\right) + \mathcal{M} + \mathcal{N}\right] \tag{5}$$

where

$$\mathcal{M} := \frac{2\lambda^2(\alpha-2)^2 + 2\lambda(3\alpha-4)}{4\lambda^2(\alpha-2)^2} \tag{6}$$

and

$$\mathcal{N} := \frac{L_1(C-D)(3\lambda-1)(3-2\lambda)}{8\lambda^2(\alpha-2)^2} \tag{7}$$

Proof. If $f \in \rho\tau\kappa^{\lambda}_{s,\Sigma}(\alpha, P(z))$, then by the definition of subordination, there exists a function w analytic in $\mathcal{U}$ with $w(0) = 0$ and $\mid w(z) < 1 \mid, z \in \mathcal{U}$, such that

$$\left(\frac{2z\left(f'(z)\right)^{\lambda}}{f(z)-f(-z)}\right)^{\alpha}\left(\frac{2\left(zf'(z)\right)^{\lambda}}{[f(z)-f(-z)]'}\right)^{1-\alpha} = \frac{(C+1)\psi(z)-(C-1)}{(D+1)\psi(z)-(D-1)}, z \in \mathcal{U}.$$

Let $l \in \psi$ be of the form $l(z) = 1 + \sum_{n=1}^{\infty} P_k\, z^k$ and it is defiend by

$$l(z) = \frac{1+w(z)}{1-w(z)}, z \in \mathcal{U}$$

On simple computation gives

$$w(z) = \frac{l(z)-1}{l(z)+1} = \frac{1}{2}p_1 z_1 + \frac{1}{2}\left(p_2 - \frac{1}{2}p_1^2\right) + \frac{1}{2}\left(p_3 - p_1 p_2 + \frac{1}{4}p_1^3\right) z^3 + \cdots, z \in \mathcal{U}$$

and considering

$$\frac{(C+1)\psi(z)-(C-1)}{(D+1)\psi(z)-(D-1)} = 1 + \frac{L_1 p_1 (C-D) z}{4} + \frac{(C-D)L_1}{4}\left[p_2 - p_1^2\left(\frac{(D+!)L_1 + 2\left(1-\frac{L_2}{L_1}\right)}{4}\right)\right] z^2 + \cdots,$$

we have

$$\begin{aligned}\left(\frac{2z\left(f'(z)\right)^{\lambda}}{f(z)-f(-z)}\right)^{\alpha}\left(\frac{2\left(zf'(z)\right)^{\lambda}}{[f(z)-f(-z)]'}\right)^{1-\alpha} &= 1 + \frac{L_1 p_1 (C-D) z}{4} \\ &+ \frac{(C-D)L_1}{4}\left[p_2 - p_1^2\left(\frac{(D+!)L_1 + 2\left(1-\frac{L_2}{L_1}\right)}{4}\right)\right] z^2 + \cdots,\end{aligned}$$

The left hand side of the above equation will be of the form

$$\begin{aligned}\left(\frac{2z\left(f'(z)\right)^{\lambda}}{f(z)-f(-z)}\right)^{\alpha}\left(\frac{2\left(zf'(z)\right)^{\lambda}}{[f(z)-f(-z)]'}\right)^{1-\alpha} &= 1 - 2\lambda(\alpha-2)a_2 z \\ &+ \left([2\lambda^2(\alpha-2)^2 + 2\lambda(3\alpha-4)]a_2^2 + (3\lambda-1)(3-2\alpha)a_3\right) z^2 + \cdots \\ &= 1 + \frac{L_1 p_1 (C-D) z}{4} \\ &+ \frac{(C-D)L_1}{4}\left[p_2 - p_1^2\left(\frac{(D+!)L_1 + 2\left(1-\frac{L_2}{L_1}\right)}{4}\right)\right] z^2 + \cdots,\end{aligned}$$

Equating the coefficients from the power series, we obtain

$$a_2 = \frac{-L_1 p_1 (C-D)}{8\lambda(\alpha-2)} \tag{8}$$

and

$$a_3 = \frac{(C-D)L_1}{4(3\lambda-1)(3-2\alpha)}\left[p_2 - p_1^2\left(\frac{L_1(D+1)}{4} + \frac{\left(1-\frac{L_2}{L_1}\right)}{2} + \left[2\lambda^2(\alpha-2)^2 + 2\lambda(3\alpha-4)\right]\frac{(C-D)L_1}{16\lambda^2(\alpha-2)^2}\right)\right] \tag{9}$$

From (2) we see that $b_2 = -a_2$ and applying Lemma 2 for (8), we obtain the inequality (2). Also, from (2) we have

$$
\begin{aligned}
b_3 &= \frac{(-1)^4}{3!}\begin{pmatrix}3a_2 & 1\\ 6a_2 & 4a_2\end{pmatrix}\\
&= 2a_2^2 - a_3\\
&= \frac{L_1^2 p_1^2 (C-D)^2}{32\lambda^2(\alpha-2)^2}\\
&\quad -\frac{(C-D)L_1}{4(3\lambda-1)(3-2\alpha)}\left[p_2 - p_1^2\left(\frac{L_1(D+1)}{4}+\frac{\left(1-\frac{L_2}{L_1}\right)}{2}+\left[2\lambda^2(\alpha-2)^2+2\lambda(3\alpha-4)\right]\frac{(C-D)L_1}{16\lambda^2(\alpha-2)^2}\right)\right]\\
&= \frac{-L_1(C-D)}{4(3\lambda-1)(3-2\lambda)}\left[p_2-\frac{1}{4}p_1^2\left(L_1(D+1)+2\left(1-\frac{L_2}{L_1}\right)+\mathcal{M}+\mathcal{N}\right)\right]
\end{aligned}
$$

where $\mathcal{M}$ and $\mathcal{N}$ are given by (6) and (7). Now using Lemma 2 we get (4), with v given by (5). □

4. Fekete-Szegö Inequalility for the Function of $\rho\tau\kappa^{\lambda}_{s,\Sigma}(\alpha, P(z))$

We will give the solution of the Fekete-Szegö problem for the functions that belong to the classes we defined in the first section.

Theorem 2. *Let $f \in \rho\tau\kappa^{\lambda}_{s,\Sigma}(\alpha, P(z))$ given by (), then for all $\mu \in \mathbb{C}$ we have*

$$
\mid a_3 - \mu a_2^2 \mid \leq \frac{\mid L_1 \mid (C-D)}{\mid 4(3\lambda-1)(3-2\lambda)\mid} max\,[1; \mid 2\tau - 1 \mid]
$$

with

$$
\tau := \frac{1}{4}\left[L_1(D+1)+2\left(1-\frac{L_2}{L_1}\right)+\mathcal{M}+\frac{\mu\mathcal{N}}{2}\right]
$$

where

$$
\mathcal{M} := \frac{2\lambda^2(\alpha-2)^2+2\lambda(3\alpha-4)}{4\lambda^2(\alpha-2)^2}
$$

and

$$
\mathcal{N} := \frac{L_1(C-D)(3\lambda-1)(3-2\lambda)}{8\lambda^2(\alpha-2)^2}
$$

the inequality is sharp for $\mu \in \mathbb{C}$

Proof. If $f \in \rho\tau\kappa^{\lambda}_{s,\Sigma}(\alpha, P(z))$, in the view of relation (8) and (9), for $\mu \in \mathbb{C}$ we have

$$
\begin{aligned}
\mid a_3 - \mu a_2^2 \mid &= \frac{(C-D)L_1}{4(3\lambda-1)(3-2\alpha)}\left[p_2 - p_1^2\left(\frac{L_1(D+1)}{4}+\frac{\left(1-\frac{L_2}{L_1}\right)}{2}\right.\right.\\
&\quad \left.\left.+\left[2\lambda^2(\alpha-2)^2+2\lambda(3\alpha-4)\right]\frac{(C-D)L_1}{16\lambda^2(\alpha-2)^2}\right)\right] - \mu\frac{L_1^2p_1^2(C-D)^2}{64\lambda^2(\alpha-2)^2}\\
&= \frac{L_1(C-D)}{4(3\lambda-1)(3-2\lambda)}\left[p_2-\frac{1}{4}p_1^2\left(L_1(D+1)+2\left(1-\frac{L_2}{L_1}\right)+\mathcal{M}+\frac{\mu\mathcal{N}}{2}\right)\right]
\end{aligned}
$$

$$\leq \frac{|L_1|(C-D)}{4|(3\lambda-1)(3-2\lambda)|}\left[2-\frac{1}{4}p_1^2\left(\left|\frac{L_2}{L_1}-L_1(D+1)-\mathcal{M}-\frac{\mu\mathcal{N}}{2}\right|-2\right)\right]$$

now if $\left|\frac{L_2}{L_1}-L_1(D+1)-\mathcal{M}-\frac{\mu\mathcal{N}}{2}\right| \leq 2$ in the above inequality we obtain

$$|a_3-\mu a_2^2| \leq \frac{|L_1|(C-D)}{4|(3\lambda-1)(3-2\lambda)|} \tag{10}$$

Further, If $\left|\frac{L_2}{L_1}-L_1(D+1)-\mathcal{M}-\frac{\mu\mathcal{N}}{2}\right| \geq 2$ in the same inequality we obtain

$$|a_3-\mu a_2^2| \leq \frac{|L_1|(C-D)}{4|(3\lambda-1)(3-2\lambda)|}\left(\left|\frac{L_2}{L_1}-L_1(D+1)-\mathcal{M}-\frac{\mu\mathcal{N}}{2}\right|\right) \tag{11}$$

An examination of the proof shows that the equality for (10) holds if $p_1 = 0$, $p_2 = 2$. Equivalently, by Lemma 3 we have $p(z^2) = p_2(z) = \frac{1+z^2}{1-z^2}$. Therefore, the extremal function of the class $\rho\tau\kappa_{s,\Sigma}^{\lambda}(\alpha, P(z))$ is given by

$$\left(\frac{2z\left(f'(z)\right)^{\lambda}}{f(z)-f(-z)}\right)^{\alpha}\left(\frac{2\left(zf'(z)\right)^{\lambda}}{[f(z)-f(-z)]'}\right)^{1-\alpha} = \frac{(C+1)p(z^2)-(C-1)}{(D+1)p(z^2)-(D-1)}.$$

Similarly, the equality for (11) holds if $p_2 = 2$. Equivalently, by Lemma 3 we have $p(z) = p_1(z) = \frac{1+z}{1-z}$. Therefore, the extremal function of the class $\rho\tau\kappa_{s,\Sigma}^{\lambda}(\alpha, P(z))$ is given by

$$\left(\frac{2z\left(f'(z)\right)^{\lambda}}{f(z)-f(-z)}\right)^{\alpha}\left(\frac{2\left(zf'(z)\right)^{\lambda}}{[f(z)-f(-z)]'}\right)^{1-\alpha} = \frac{(C+1)p_1(z)-(C-1)}{(D+1)p_1(z)-(D-1)}.$$

and the proof of the theorem is complete. □

5. Conclusion

We unify and extend various classes of analytic function by defining λ-pseudo starlike function using subordination. Also several results which are closely related to the results presented here, refer to [11, 12] and references provided therein.

Declarations

Acknowledgements: The author would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions

Author's Contribution: The author, M.I., contributed to this manuscript fully in theoretic and structural points.

Conflict of Interest Disclosure: The author declares no conflict of interest.

Supporting/Supporting Organizations: This research received no external funding.

Ethical Approval and Participant Consent: This article does not contain any studies with human or animal subjects. It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of Data and Materials: Data sharing not applicable.

Use of AI tools: The author declares that they have not used Artificial Intelligence (AI) tools in the creation of this article.

ORCID

Musthafa Ibrahim https://orcid.org/0000-0003-3622-0358

References

[1] W.C. Ma and D. Minda, *A unified treatment of some special classes of univalent functions*, In Proceedings of the Conference on Complex Analysis, Tianjin, China, 19–23 June 1992; Li, Z., Ren, F., Yang, L., Zhang, S., Eds., Conference Proceedings and Lecture Notes in Analysis, International Press: Cambridge, MA, USA, **1** (1994), 157–169. [Web of Science]

[2] O.T. Opoola, *On a subclass of univalent functions defined by a generalized differential operator*, Int. J. Math. Anal., **18**(11) (2017), 869-876. [CrossRef]

[3] L. Bieberbach, *Über dié Koeffizienten derjenigen Potenzreihen*, welche eine schlichte Abbildung des Einheitskreises vermitteln. Sitzungsberichte Preuss. Akad. Der Wiss. 138 (1916), 940–955.

[4] K.O. Babalola, *On λ-pseudo-starlike functions*, J. Class. Anal., **3**(2) (2013), 137-147. [CrossRef]

[5] V. Ravichandran, *Starlike and convex functions with respect to conjugate points*, Acta Math. Acad. Paedagog. Nyházi. (N.S.), **20**(1) (2004), 31-37. MR2129665 [Web]

[6] K.B. Sakaguchi, *On a certain univalent mapping*, J. Math. Soc. Japan, **11**(1) (1959), 72-75. [CrossRef] [Scopus]

[7] Z.G. Wang, C.Y. Gao and S.M. Yuan, *On certain subclasses of close-to-convex and quasi-convex functions with respect to k-symmetric points*, J. Math. Anal. Appl., **322**(1) (2006), 97-106. [CrossRef] [Scopus] [Web of Science]

[8] K.R. Karthikeyan, M. Ibrahim and S. Srinivasan, *Fractional class of analytic functions defined using q-differential operator*, Aust. J. Math. Anal. Appl., **15**(1) (2018), 1-15. [Scopus]

[9] A.W. Goodman, *Univalent Functions*, Mariner Comp., (1983).

[10] C. Pommerenke, *Univalent Functions*, Studia Mathematica/Mathematische Lehrbücher, Band XXV, Vandenhoeck & Ruprecht, Göttingen, (1975).

[11] H. Tang, K.R. Karthikeyan and G. Murugusundaramoorthy, *Certain subclass of analytic functions with respect to symmetric points associated with conic region*, AIMS Math., **6**(11) (2021), 12863-12877. [CrossRef] [Scopus] [Web of Science]

[12] M. Ibrahim and K.R Karthikeyan, *Unified solution of some properties related to λ-pseudo starlike functions*, Contemp. Math., **4**(4) (2023), 926-936. [CrossRef] [Scopus] [Web of Science]

Advances in Analysis and Applied Mathematics (AAAM), (Adv. Anal. Appl. Math.)
https://advmath.org/index.php/pub

How to cite this article: M. Ibrahim, *Subclasses of λ-pseudo Starlike functions with respect to symmetric points associated with conic region*, Adv. Anal. Appl. Math., **1**(1) (2024), 12-18. DOI 10.62298/advmath.5