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Research Paper

Comprehensive Subfamilies of Bi-Univalent Functions Defined by Error Function Subordinate to Gegenbauer Polynomials

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Abstract

This work explores coefficient estimates for analytic functions in a symmetric domain. Building on recent studies that use polynomials such as Lucas and Legendre polynomials to bound the Maclaurin coefficients $|a_2|$ and $|a_3|$, we turn to Gegenbauer polynomials. By applying the imaginary error function and subordination techniques, we derive sharp bounds for $|a_2|$ and $|a_3|$ and the Fekete-Szegő functional for two extensive new subclasses. Our general theorems also yield several novel special cases, demonstrating the breadth of our results.

Key Words: Analytic; Univalent; Bi-univalent; Symmetric domain; Error function, Gegenbauer polynomials; Fekete-Szegő.
AMS 2020 Classification: 05A30; 30C45; 11B65; 47B38

1. Introduction

The investigation of bi-univalent functions via error functions integrates advanced mathematical techniques, error assessment, and approximation with complex analysis, particularly function theory. Bi-univalent functions refer to subclasses of univalent functions that exhibit analyticity within a specific domain. The application of error functions to bi-univalent functions is driven by the convergence of classical function theory, numerical analysis, and practical implications in engineering and physics. Leveraging error functions enhances our understanding of bi-univalent functions by providing more precise descriptions, tighter bounds, and improved approximations. Furthermore, the error function has diverse applications in probability theory, statistical analysis, and partial differential equations. Notably, in quantum mechanics, the error function plays a vital role in estimating the probability of locating a particle within a defined region. Previous studies, such as those by Alzer [1] and Coman [2], have explored the various properties and inequalities of the error function, while Elbert et al. [3] have examined the characteristics of the complementary error function.

Let $\mathcal{A}$ denote the family of analytic and univalent functions f in the symmetric domain $U = \{z \in \mathbb{C} : |z| < 1\}$ and satisfying $f(0) = f'(0) - 1 = 0$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1}$$

Every function $f \in \mathcal{A}$ has an inverse f^{-1} defined by

$$f^{-1}(f(z)) = z, \qquad w = f^{-1}(f(w)), \quad (z \in U, |w| < r_0(f) \geq \frac{1}{4}),$$

where

$$f^{-1}(w) = H(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (a_4 + 5a_2^3 - 5a_3 a_2)w^4 + \cdots. \tag{2}$$

Let Σ be the family of bi-univalent functions in U given by (1) (i.e., f is bi-univalent in U if both f and f^{-1} are univalent in U); see [5]. The function f is subordinate to H, denoted by $f \prec H$, if there exists a function $w \in \mathcal{A}$ such that both f and H are analytic in U and

$$w(0) = 0, \qquad |w(z)| < 1, \quad (z \in U)$$

such that

$$f(z) = H(w(z)).$$

Abramowitz and Stegun [6] defined the following error function:

$$erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)k!}, \quad (z \in \mathbb{C}). \tag{3}$$

Further, we define the imaginary error function, denoted by $erfi$, as follows:

$$erfi(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{t^2} dt = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)k!}, \quad (z \in \mathbb{C}). \tag{4}$$

Since the error function is odd, i.e., $erf(-z) = -erf(z)$, it is symmetric with respect to the origin. The generalized form of (3) can be written as:

$$\begin{aligned} erf_{\mu}(z) &= \frac{\mu!}{\sqrt{\pi}} \int_0^z e^{-t^{\mu}} dt, \quad \mu \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \\ &= \frac{\mu}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k z^{\mu k+1}}{(\mu k+1)k!}, \quad (z \in \mathbb{C}). \end{aligned} \tag{5}$$

From (5), we have $erf_0(z) = \frac{z}{\sqrt[e]{\pi}}$, $erf_1(z) = \frac{1-e^z}{\sqrt{\pi}}$, $erf_2(z) = erf(z)$. Clearly, the function $erf_{\mu}(z)$ does not belong to the family $\mathcal{A}$. Now we consider the following function:

$$\varepsilon_{\mu}(z) = \frac{\sqrt{\pi}}{\mu!} z^{\left(1 - \frac{1}{\mu}\right)} erf_{\mu}\left(z^{\frac{1}{\mu}}\right)$$

$$= \quad z+\sum_{k=2}^{\infty}\frac{(-1)^{k-1}}{((k-1)\mu+1)(k-1)!}z^{k}, \quad (\mu\in\mathbb{N}, z\in\mathbb{C}). \tag{6}$$

Also, the imaginary error function (4) is generalized as follows:

$$erfi_{\mu}(z)=\frac{\mu}{\sqrt{\pi}}\int_{0}^{z}e^{t^{\mu}}dt=\frac{\mu!}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{z^{\mu k+1}}{(\mu k+1)k!}, \quad (\mu\in\mathbb{N}_0, z\in\mathbb{C}). \tag{7}$$

Further, the normalization of the generalized imaginary error function $erfi_{\mu}(z)$ is given by

$$\begin{aligned} E_{\mu}(z) &= \frac{\sqrt{\pi}}{\mu!}z^{\left(1-\frac{1}{\mu}\right)}erfi_{\mu}\left(z^{\frac{1}{\mu}}\right) \\ &= z+\sum_{k=2}^{\infty}\frac{1}{((k-1)\mu+1)(k-1)!}z^{k}, \quad (\mu\in\mathbb{N}, z\in\mathbb{C}). \end{aligned} \tag{8}$$

Making use of the convolution, we construct the linear operator $Ef_{\mu}(z):\mathcal{A}\to\mathcal{A}$ given by

$$Ef_{\mu}(z)=f(z)*E_{\mu}(z)=z+\sum_{k=2}^{\infty}\frac{1}{((k-1)\mu+1)(k-1)!}a_k z^{k}. \tag{9}$$

Remark 1. If we take $\mu=2$ in (6), we obtain the normalization for Ramachandran et al. [7], and if we take $\mu=2$ in (8), we obtain the normalization for Mohammed et al. [8].

The problem of estimating the coefficient for each of $|a_n|$ $(n\geq 3; n\in\mathbb{N})$ is presumably still an open problem. Brannan and Taha [4] presented the subclasses of the class of bi-univalent functions Σ, namely $S_{\Sigma}^{*}(\alpha)$ and $K_{\Sigma}(\alpha)$ of bi-starlike and bi-convex functions of order α $(0<\alpha\leq 1)$, and the first two coefficients were estimated. These results are similar to the well-known subclasses $S^{*}(\alpha)$ and $K(\alpha)$ of starlike and convex functions of order α $(0\leq\alpha<1)$. Additional examples and details related to the class Σ can be found in references [5, 6, 7, 8].

Orthogonal polynomials have been widely studied since their discovery by Legendre in 1784 [9]. They have been used as a mathematical approach to solve ordinary differential equations associated with model problems under certain conditions. The advantages of orthogonal polynomials in modern mathematics and their application in physics and engineering cannot be ignored. In mathematics, orthogonal polynomials play a key role in approximation theory, differential and integral equations, and mathematical statistics. Additionally, these polynomials have been instrumental in various applications, such as scattering theory, quantum mechanics, signal analysis, automatic control, and axially symmetric potential theory [10, 11].

Amourah et al. [12] investigated Gegenbauer polynomials, whose generating function $H_{\alpha}(x,z)$ is given by

$$H_{\alpha}(x,z)=\frac{1}{(1-2xz+z^2)^{\alpha}}, \tag{10}$$

where $-1\leq x\leq 1$ and $z\in U$. Also, since H_{α} is analytic in U, it can be written in a power series expansion as follows:

$$H_{\alpha}(x,z)=\sum_{k=0}^{\infty}C_n^{\alpha}(x)z^n,$$

where $C_n^{\alpha}(x)$ is a Gegenbauer polynomial of degree n. The Gegenbauer polynomials generate Legendre polynomials and Chebyshev polynomials when setting α as $1/2$ and 1, respectively, and they can also be defined by the following recurrence relations:

$$C_n^{\alpha}(x) = \frac{1}{n}[2x(n+\alpha-1)C_{n-1}^{\alpha}(x) - (n+2\alpha-2)C_{n-2}^{\alpha}(x)],$$

with the initial values:

$$\begin{aligned} C_0^{\alpha}(x) &= 1, \\ C_1^{\alpha}(x) &= 2\alpha x, \\ C_2^{\alpha}(x) &= 2\alpha(1+\alpha)x^2 - \alpha. \end{aligned} \tag{11}$$

In this work, we construct two new and extensive subfamilies of bi-univalent functions using a particular special function, the imaginary error function and Gegenbauer polynomials, denoted by $G_{\Sigma}^{\alpha}(\zeta, \epsilon, x)$ and $T_{\Sigma}^{\alpha}(\varphi, x)$, and find initial bounds for the coefficients $|a_2|$ and $|a_3|$, as well as the Fekete-Szegő inequality. Also, a number of new corollaries are presented.

2. Bounds of the Subfamilies $G_{\Sigma}^{\alpha}(\zeta, \epsilon, x)$ and $T_{\Sigma}^{\alpha}(\varphi, x)$

At the beginning of this section, we define the comprehensive subfamilies $G_{\Sigma}^{\alpha}(\zeta, \epsilon, x)$ and $T_{\Sigma}^{\alpha}(\varphi, x)$ using an error function subordinate to Gegenbauer polynomials.

Definition 1. For $f \in G_{\Sigma}^{\alpha}(\zeta, \epsilon, x)$, assume that the following subordinations are satisfied:

$$(1-\zeta)\frac{Ef_{\mu}(z)}{z} + \zeta(Ef_{\mu}(z))' + \epsilon z(Ef_{\mu}(z))'' \prec H_{\alpha}(x, z), \tag{12}$$

$$(1-\zeta)\frac{EH_{\mu}(w)}{w} + \zeta(EH_{\mu}(w))' + \epsilon w(EH_{\mu}(w))'' \prec H_{\alpha}(x, w), \tag{13}$$

where $\zeta \geq 1$, $\epsilon \geq 0$, $\frac{1}{2} < x < 1$, $z, w \in U$, and $H = f^{-1}$.

Definition 2. For $f \in T_{\Sigma}^{\alpha}(\varphi, x)$, assume that the following subordinations are satisfied:

$$(Ef_{\mu}(z))' + z\frac{e^{i\varphi}+1}{2}(Ef_{\mu}(z))'' \prec H_{\alpha}(x, z), \tag{14}$$

$$(EH_{\mu}(w))' + w\frac{e^{i\varphi}+1}{2}(EH_{\mu}(w))'' \prec H_{\alpha}(x, w), \tag{15}$$

where $-\pi < \varphi \leq \pi$, $\frac{1}{2} < x \leq 1$, $z, w \in U$, and $H = f^{-1}$.

Example 1. If we put $\zeta = 1$ in Definition 1, we obtain the subfamily $G_{\Sigma}^{\alpha}(1, \epsilon, x)$, which satisfies the following requirements:

$$(Ef_{\mu}(z))' + \epsilon z(Ef_{\mu}(z))'' \prec H_{\alpha}(x, z),$$

$$(EH_{\mu}(w))' + \epsilon w(EH_{\mu}(w))'' \prec H_{\alpha}(x, w),$$

where $\epsilon \geq 0$, $\frac{1}{2} < x \leq 1$, $z, w \in U$, and $H = f^{-1}$.

Example 2. If $\epsilon = 0$ in Definition 1, we obtain the subfamily $G_{\Sigma}^{\alpha}(\zeta, 0, x)$, which satisfies the following requirements:

$$(1-\zeta)\frac{Ef_{\mu}(z)}{z} + \zeta(Ef_{\mu}(z))' \prec H_{\alpha}(x, z),$$

$$(1-\zeta)\frac{EH_{\mu}(w)}{w} + \zeta(EH_{\mu}(w))' \prec H_{\alpha}(x, w),$$

where $\zeta \geq 1$, $\frac{1}{2} < x \leq 1$, $z, w \in U$, and $H = f^{-1}$.

Example 3. If $\varphi = \pi$ in Definition 2, we obtain the subfamily $T_{\Sigma}^{\alpha}(\pi, x)$, which satisfies the following requirements:

$$(Ef_{\mu}(z))' \prec H_{\alpha}(x, z)$$

and

$$(EH_{\mu}(w))' \prec H_{\alpha}(x, w),$$

where $\frac{1}{2} < x \leq 1$, $z, w \in U$, and $H = f^{-1}$.

Example 4. If $\varphi = 0$ in Definition 2, we obtain the subfamily $T_{\Sigma}^{\alpha}(0, x)$, which satisfies the following requirements:

$$(Ef_{\mu}(z))' + z\,(Ef_{\mu}(z))'' \prec H_{\alpha}(x, z),$$

$$(EH_{\mu}(w))' + w\,(EH_{\mu}(w))'' \prec H_{\alpha}(x, w),$$

where $\frac{1}{2} < x \leq 1$, $z, w \in U$, and $H = f^{-1}$.

Remark 2. All the previous subfamilies mentioned are inspired by the subfamilies used by many researchers when $\mathrm{Re}(f'(z)) > \alpha$. From this, we can determine $\mathrm{Re}(f'(z)) > 0$, which is the condition for the function f to be univalent on the open disk U.

Lemma 1. ([13]). Let $X(z) \in F$ be given by

$$X(z) = 1 + m_1 z + m_2 z^2 + m_3 z^3 + \cdots, \quad (\mathrm{Re}\,(X(z)) > 0, z \in U).$$

Then

$$|m_n| \leq 2$$

for each $n \in \mathbb{N}$.

In the next theorem, we estimate the initial coefficients $|a_2|$ and $|a_3|$ and solve the Fekete-Szegő problems for the subfamilies $G_{\Sigma}^{\alpha}(\zeta, \epsilon, x)$ and $T_{\Sigma}^{\alpha}(\varphi, x)$, respectively.

3. Main Results

We begin by estimating the upper bound of the coefficients for the functions belonging to class $f \in \mathcal{R}_q(h)$.

Theorem 1. *Let $f \in \Sigma$ be given by (1) in the subfamily $G_{\Sigma}^{\alpha}(\zeta, \epsilon, x)$, where $\zeta \geq 1$, $\epsilon \geq 0$, $\frac{1}{2} < x \leq 1$, $z, w \in U$, and $H = f^{-1}$. Then*

$$|a_2| \leq \sqrt{D(\epsilon, \zeta, x)},$$

$$|a_3| \leq \frac{4\alpha^2 x^2 (\mu+1)^2}{(2\epsilon+\zeta+1)^2} + \frac{4\alpha x\,(2\mu+1)}{(6\epsilon+2\zeta+1)},$$

and

$$\left|a_3 - \theta a_2^2\right| \leq \begin{cases} \dfrac{8\alpha x(2\mu+1)}{(6\epsilon+2\zeta+1)}, & \text{if } (1-\theta)D(\epsilon,\zeta,x) < \dfrac{4\alpha x(2\mu+1)}{(6\epsilon+2\zeta+1)}, \\[2ex] 2(1-\theta)D(\epsilon,\zeta,x), & \text{if } (1-\theta)D(\epsilon,\zeta,x) \geq \dfrac{4\alpha x(2\mu+1)}{(6\epsilon+2\zeta+1)}, \end{cases}$$

where

$$D(\epsilon, \zeta, x) = \frac{8\alpha^3 x^3 (2\mu+1)(\mu+1)^2}{2(6\epsilon+2\zeta+1)(\mu+1)\,\alpha^2 x^2 - (2\epsilon+\zeta+1)^2(2\mu+1)(2\alpha(1+\alpha)x^2 - (1+2\alpha)x)}.$$

Proof. Since $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in G_{\Sigma}^{\alpha}(\zeta, \epsilon, x)$, from (12) and (13), we have

$$(1-\zeta)\frac{Ef_{\mu}(z)}{z} + \zeta(Ef_{\mu}(z))' + \epsilon z(Ef_{\mu}(z))'' \prec H_{\alpha}(x,z) \tag{16}$$

and

$$(1-\zeta)\frac{EH_{\mu}(w)}{w} + \zeta(EH_{\mu}(w))' + \epsilon w(EH_{\mu}(w))'' \prec H_{\alpha}(x,w). \tag{17}$$

We define the functions $s_1, s_2 : U \to U$ with $s_1(0) = s_2(0) = 0$ and $|s_1(z)| < 1$, $|s_2(w)| < 1$ for all $z, w \in U$. So we can define $\rho, \sigma \in F$ as

$$\rho(z) = \frac{1+s_1(z)}{1-s_1(z)} = 1 + \rho_1 z + \rho_2 z^2 + \rho_3 z^3 + \cdots, \quad |\rho_k| \leq 2, \quad z \in U.$$

Then

$$s_1(z) = \frac{\rho(z)-1}{\rho(z)+1} = \frac{\rho_1}{2}z + \left(\frac{\rho_2}{2} - \frac{\rho_1^2}{4}\right)z^2 + \frac{1}{2}\left(\rho_3 - \rho_1\rho_2 + \frac{\rho_1^3}{4}\right)z^3 + \cdots. \tag{18}$$

Similarly,

$$\sigma(w) = \frac{1+s_2(w)}{1-s_2(w)} = 1 + \sigma_1 w + \sigma_2 w^2 + \sigma_3 w^3 + \cdots, \quad |\sigma_k| \leq 2, \quad w \in U,$$

and

$$s_2(w) = \frac{\sigma(w)-1}{\sigma(w)+1} = \frac{\sigma_1}{2}w + \left(\frac{\sigma_2}{2} - \frac{\sigma_1^2}{4}\right)w^2 + \frac{1}{2}\left(\sigma_3 - \sigma_1\sigma_2 + \frac{\sigma_1^3}{4}\right)w^3 + \cdots. \tag{19}$$

Using (18) and (19) in (11), we obtain

$$\begin{aligned} H_{\alpha}(x, s_1(z)) &= C_0^{\alpha}(x) + C_1^{\alpha}(x)\frac{\rho_1}{2}z + \left(C_1^{\alpha}(x)\left(\frac{\rho_2}{2} - \frac{\rho_1^2}{4}\right)\right. \\ &\quad \left. + C_2^{\alpha}(x)\frac{\rho_1^2}{4}\right)z^2 + \left(\frac{1}{2}C_1^{\alpha}(x)\left(\rho_3 - \rho_1\rho_2 + \frac{\rho_1^3}{4}\right)\right. \\ &\quad \left. + C_2^{\alpha}(x)\left(\frac{\rho_1\rho_2}{2} - \frac{\rho_1^3}{4}\right) + C_3^{\alpha}(x)\frac{\rho_1^3}{8}\right)z^3 + \cdots, \end{aligned} \tag{20}$$

and similarly,

$$\begin{aligned} H_{\alpha}(x, s_2(w)) &= C_0^{\alpha}(x) + C_1^{\alpha}(x)\frac{\sigma_1}{2}w + \left(C_1^{\alpha}(x)\left(\frac{\sigma_2}{2} - \frac{\sigma_1^2}{4}\right)\right. \\ &\quad \left. + C_2^{\alpha}(x)\frac{\sigma_1^2}{4}\right)w^2 + \left(\frac{1}{2}C_1^{\alpha}(x)\left(\sigma_3 - \sigma_1\sigma_2 + \frac{\sigma_1^3}{4}\right)\right. \\ &\quad \left. + C_2^{\alpha}(x)\left(\frac{\sigma_1\sigma_2}{2} - \frac{\sigma_1^3}{4}\right) + C_3^{\alpha}(x)\frac{\sigma_1^3}{8}\right)w^3 + \cdots. \end{aligned} \tag{21}$$

From (16), (17), (20), and (21), we have

$$\frac{2\epsilon+\zeta+1}{\mu+1}a_2 = \frac{C_1^{\alpha}(x)}{2}\rho_1, \tag{22}$$

$$\frac{6\epsilon+2\zeta+1}{2(2\mu+1)}a_3 = C_1^{\alpha}(x)\left(\frac{\rho_2}{2}-\frac{\rho_1^2}{4}\right)+C_2^{\alpha}(x)\frac{\rho_1^2}{4}, \tag{23}$$

$$-\frac{2\epsilon+\zeta+1}{\mu+1}a_2 = \frac{C_1^{\alpha}(x)}{2}\sigma_1, \tag{24}$$

and

$$\frac{6\epsilon+2\zeta+1}{2(2\mu+1)}\left(2a_2^2-a_3\right) = C_1^{\alpha}(x)\left(\frac{\sigma_2}{2}-\frac{\sigma_1^2}{4}\right)+C_2^{\alpha}(x)\frac{\sigma_1^2}{4}. \tag{25}$$

Upon adding (22) and (24), we obtain

$$\rho_1 = -\sigma_1 \tag{26}$$

and

$$\frac{8(2\epsilon+\zeta+1)^2}{(\mu+1)^2}a_2^2 = C_1^{\alpha}(x)^2(\rho_1^2+\sigma_1^2). \tag{27}$$

Therefore,

$$a_2^2 = \frac{C_1^{\alpha}(x)^2(\rho_1^2)(\mu+1)^2}{4(2\epsilon+\zeta+1)^2}. \tag{28}$$

Adding (23) and (25), we get

$$\frac{4(6\epsilon+2\zeta+1)}{(2\mu+1)}a_2^2 = 2C_1^{\alpha}(x)(\rho_2+\sigma_2)+(\rho_1^2+\sigma_1^2)\left(C_2^{\alpha}(x)-C_1^{\alpha}(x)\right).$$

Using (26),

$$\frac{4(6\epsilon+2\zeta+1)}{(2\mu+1)}a_2^2 = 2C_1^{\alpha}(x)(\rho_2+\sigma_2)+2\rho_1^2\left(C_2^{\alpha}(x)-C_1^{\alpha}(x)\right).$$

Hence,

$$\frac{2(6\epsilon+2\zeta+1)}{(2\mu+1)}a_2^2 = C_1^{\alpha}(x)(\rho_2+\sigma_2)+\rho_1^2\left(C_2^{\alpha}(x)-C_1^{\alpha}(x)\right). \tag{29}$$

From (26) and (27), we obtain

$$\rho_1^2 = \frac{4(2\epsilon+\zeta+1)^2 a_2^2}{C_1^{\alpha}(x)^2(\mu+1)^2}. \tag{30}$$

By replacing (30) in (29), we obtain the following result:

$$a_2^2 = \frac{C_1^{\alpha}(x)^3(\rho_2+\sigma_2)(2\mu+1)(\mu+1)^2}{2\left[(6\epsilon+2\zeta+1)(\mu+1)C_1^{\alpha}(x)^2-2(2\epsilon+\zeta+1)^2(2\mu+1)(C_2^{\alpha}(x)-C_1^{\alpha}(x))\right]}.$$

Applying (11) and Lemma 1, we obtain

$$|a_2| \le \sqrt{\frac{8\alpha^3x^3(2\mu+1)(\mu+1)^2}{2(6\epsilon+2\zeta+1)(\mu+1)\alpha^2x^2-(2\epsilon+\zeta+1)^2(2\mu+1)(2\alpha(1+\alpha)x^2-(1+2\alpha)x)}}$$

$$= \sqrt{D(\epsilon,\zeta,x)},$$

where

$$D(\epsilon,\zeta,x) = \frac{8\alpha^3x^3(2\mu+1)(\mu+1)^2}{2(6\epsilon+2\zeta+1)(\mu+1)\alpha^2x^2-(2\epsilon+\zeta+1)^2(2\mu+1)(2\alpha(1+\alpha)x^2-(1+2\alpha)x)}.$$

Subtracting (25) from (23) and considering (26), we obtain

$$a_3 = a_2^2 + \frac{C_1^\alpha(x)(\rho_2 - \sigma_2)(2\mu+1)}{2(6\epsilon+2\zeta+1)}. \tag{31}$$

Substituting the value of a_2^2 from (28) and using (26), we have

$$a_3 = \frac{C_1^\alpha(x)^2\rho_1^2(\mu+1)^2}{4(2\epsilon+\zeta+1)^2} + \frac{C_1^\alpha(x)(\rho_2-\sigma_2)(2\mu+1)}{2(6\epsilon+2\zeta+1)}.$$

Applying (11) and Lemma 1, we obtain

$$|a_3| \le \frac{C_1^\alpha(x)^2\left|\rho_1^2\right|(\mu+1)^2}{4(2\epsilon+\zeta+1)^2} + \frac{C_1^\alpha(x)(|\rho_2|+|\sigma_2|)(2\mu+1)}{2(6\epsilon+2\zeta+1)}.$$

Thus,

$$|a_3| \le \frac{4\alpha^2x^2(\mu+1)^2}{(2\epsilon+\zeta+1)^2} + \frac{4\alpha x(2\mu+1)}{(6\epsilon+2\zeta+1)}.$$

From (31), we have

$$a_3 - \theta a_2^2 = \frac{C_1^\alpha(x)(\rho_2-\sigma_2)(2\mu+1)}{2(6\epsilon+2\zeta+1)} + (1-\theta)\,a_2^2.$$

Using (11) after the triangular inequality, we arrive at

$$\begin{aligned}\left|a_3 - \theta a_2^2\right| &\le \frac{C_1^\alpha(x)(|\rho_2|+|\sigma_2|)(2\mu+1)}{2(6\epsilon+2\zeta+1)} + |(1-\theta)|\left|a_2^2\right| \\ &\le \frac{4\alpha x(2\mu+1)}{(6\epsilon+2\zeta+1)} + (1-\theta)\,D(\epsilon,\zeta,x).\end{aligned}$$

Now, if

$$(1-\theta)\,D(\epsilon,\zeta,x) < \frac{4\alpha x(2\mu+1)}{(6\epsilon+2\zeta+1)},$$

then

$$\left|a_3-\theta a_2^2\right| \le \frac{8\alpha x(2\mu+1)}{(6\epsilon+2\zeta+1)}.$$

And if

$$(1-\theta)\,D(\epsilon,\zeta,x) \ge \frac{4\alpha x(2\mu+1)}{(6\epsilon+2\zeta+1)},$$

then we have

$$\left|a_3-\theta a_2^2\right| \le 2(1-\theta)\,D(\epsilon,\zeta,x).$$

Hence proved. □

Theorem 2. *Let $f \in \Sigma$ be given by (1) in the subfamily $T_\Sigma^\alpha(\varphi, x)$, where $-\pi < \varphi \le \pi$, $\epsilon \ge 0$, $\frac{1}{2} < x \le 1$, $z, w \in U$, and $H = f^{-1}$. Then*

$$|a_2| \le \sqrt{Y(\varphi,x)},$$

$$|a_3| \le \frac{4\alpha^2x^2(\mu+1)^2}{(e^{i\varphi}+3)^2} + \frac{4\alpha x(2\mu+1)}{3(e^{i\varphi}+2)},$$

and

$$\left|a_3-\theta a_2^2\right| \le \begin{cases} \dfrac{8\alpha x(2\mu+1)}{3(e^{i\varphi}+2)}, & \text{if } 0 \le (1-\theta)\,Y(\varphi,x) < \dfrac{4\alpha x(2\mu+1)}{3(e^{i\varphi}+2)}, \\ 2(1-\theta)\,Y(\varphi,x), & \text{if } (1-\theta)\,Y(\varphi,x) \ge \dfrac{4\alpha x(2\mu+1)}{3(e^{i\varphi}+2)}, \end{cases}$$

where

$$Y(\varphi,x) = \frac{2(2\mu+1)(\mu+1)^2\alpha^3x^3}{3\left[\alpha^2x^2(e^{i\varphi}+2)(\mu+1) - (e^{i\varphi}+3)^2(2\mu+1)(2\alpha(1+\alpha)x^2-(1+2\alpha)x)\right]}.$$

Proof. Since $f(z)=z+\sum\limits_{k=2}^{\infty}a_kz^k\in T_{\Sigma}^{\alpha}(\varphi,x)$, from (14), (15), (20), and (21), we can write

$$(Ef_{\mu}(z))'+z\frac{e^{i\varphi}+1}{2}(Ef_{\mu}(z))''\prec H_{\alpha}(x,z) \tag{32}$$

and

$$(EH_{\mu}(w))'+w\frac{e^{i\varphi}+1}{2}(EH_{\mu}(w))''\prec H_{\alpha}(x,w). \tag{33}$$

From (32) and (33) and the functions $H_{\alpha}(x,z)$ and $H_{\alpha}(x,w)$, respectively, which are given by (20) and (21), we have

$$\frac{e^{i\varphi}+3}{(\mu+1)}a_2=\frac{C_1^{\alpha}(x)}{2}\rho_1, \tag{34}$$

$$\frac{3\left(e^{i\varphi}+2\right)}{2\left(2\mu+1\right)}a_3=C_1^{\alpha}(x)\left(\frac{\rho_2}{2}-\frac{\rho_1^2}{4}\right)+C_2^{\alpha}(x)\frac{\rho_1^2}{4}, \tag{35}$$

$$-\frac{e^{i\varphi}+3}{(\mu+1)}a_2=\frac{C_1^{\alpha}(x)}{2}\sigma_1, \tag{36}$$

and

$$\frac{3\left(e^{i\varphi}+2\right)}{2\left(2\mu+1\right)}(2a_2^2-a_3)=C_1^{\alpha}(x)\left(\frac{\sigma_2}{2}-\frac{\sigma_1^2}{4}\right)+C_2^{\alpha}(x)\frac{\sigma_1^2}{4}. \tag{37}$$

We obtain the findings provided by Theorem 2 using the same method to prove Theorem 1. □

3.1. Set of corollaries

Corollary 1. Let $f\in G_{\Sigma}^{\alpha}(1,\epsilon,x)$, where $\epsilon\geq 0$, $\frac{1}{2}<x\leq 1$, $z,w\in U$. Then

$$|a_2|\leq\sqrt{D(\epsilon,1,x)},$$

$$|a_3|\leq\frac{\alpha^2x^2(\mu+1)^2}{(\epsilon+1)^2}+\frac{4\alpha x(2\mu+1)}{3\left(2\epsilon+1\right)},$$

and

$$\left|a_3-\theta a_2^2\right|\leq\begin{cases}\dfrac{8\alpha x(2\mu+1)}{3(2\epsilon+1)}, & \text{if } (1-\theta)D(\epsilon,1,x)<\dfrac{4\alpha x(2\mu+1)}{3(2\epsilon+1)},\\[2ex] 2(1-\theta)D(\epsilon,1,x), & \text{if } (1-\theta)D(\epsilon,1,x)\geq\dfrac{4\alpha x(2\mu+1)}{3(2\epsilon+1)},\end{cases}$$

where

$$D(\epsilon,1,x)=\frac{\alpha^3x^3(2\mu+1)(\mu+1)^2}{3(2\epsilon+1)(\mu+1)\alpha^2x^2-2(\epsilon+1)^2(2\mu+1)(2\alpha(1+\alpha)x^2-(1+2\alpha)x)}.$$

Corollary 2. Let $f \in G_{\Sigma}^{\alpha}(\zeta, 0, x)$, where $\zeta \geq 1$, $\frac{1}{2} < x \leq 1$, $z, w \in U$. Then

$$|a_2| \leq \sqrt{D\left(0, \zeta, x\right)},$$

$$|a_3| \leq \frac{4\alpha^2 x^2 \left(\mu+1\right)^2}{(\zeta+1)^2} + \frac{4\alpha x \left(2\mu+1\right)}{\left(2\zeta+1\right)},$$

and

$$\left|a_3 - \theta a_2^2\right| \leq \begin{cases} \dfrac{8\alpha x(2\mu+1)}{(2\zeta+1)}, & \text{if } (1-\theta)D(0,\zeta,x) < \dfrac{4\alpha x(2\mu+1)}{(2\zeta+1)}, \\ 2(1-\theta)D(0,\zeta,x), & \text{if } (1-\theta)D(0,\zeta,x) \geq \dfrac{4\alpha x(2\mu+1)}{(2\zeta+1)}, \end{cases}$$

where

$$D(0,\zeta,x) = \frac{\alpha^3 x^3 \left(2\mu+1\right)\left(\mu+1\right)^2}{\left(2\zeta+1\right)\left(\mu+1\right)\alpha^2 x^2 - (\zeta+1)^2(2\mu+1)(2\alpha(1+\alpha)x^2 - (1+2\alpha)x)}.$$

Corollary 3. Let $f \in T_{\Sigma}^{\alpha}(\pi, x)$, where $\frac{1}{2} < x \leq 1$, $z, w \in U$. Then

$$|a_2| \leq \sqrt{Y\left(\pi, x\right)},$$

$$|a_3| \leq \alpha^2 x^2 \left(\mu+1\right)^2 + \frac{4\alpha x(2\mu+1)}{3},$$

and

$$\left|a_3 - \theta a_2^2\right| \leq \begin{cases} \dfrac{8\alpha x(2\mu+1)}{3}, & \text{if } 0 \leq \left(1-\theta\right) Y\left(\pi, x\right) < \dfrac{4\alpha x(2\mu+1)}{3}, \\ 2\left(1-\theta\right) Y\left(\pi, x\right), & \text{if } \left(1-\theta\right) Y\left(\pi, x\right) \geq \dfrac{4\alpha x(2\mu+1)}{3}, \end{cases}$$

where

$$Y\left(\pi, x\right) = \frac{2\left(2\mu+1\right)\left(\mu+1\right)^2 \alpha^3 x^3}{3\left[\alpha^2 x^2\left(\mu+1\right) - 4\left(2\mu+1\right)\left(2\alpha(1+\alpha)x^2 - (1+2\alpha)x\right)\right]}.$$

Corollary 4. Let $f \in T_{\Sigma}^{\alpha}(0, x)$, where $\frac{1}{2} < x \leq 1$, $z, w \in U$. Then

$$|a_2| \leq \sqrt{Y\left(0, x\right)},$$

$$|a_3| \leq \frac{\alpha^2 x^2 \left(\mu+1\right)^2}{4} + \frac{4\alpha x(2\mu+1)}{9},$$

and

$$\left|a_3 - \theta a_2^2\right| \leq \begin{cases} \dfrac{8\alpha x(2\mu+1)}{9}, & \text{if } 0 \leq \left(1-\theta\right) Y\left(0, x\right) < \dfrac{4\alpha x(2\mu+1)}{9}, \\ 2\left(1-\theta\right) Y\left(0, x\right), & \text{if } \left(1-\theta\right) Y\left(0, x\right) \geq \dfrac{4\alpha x(2\mu+1)}{9}, \end{cases}$$

where

$$Y\left(0, x\right) = \frac{2\left(2\mu+1\right)\left(\mu+1\right)^2 \alpha^3 x^3}{3\left[3\alpha^2 x^2\left(\mu+1\right) - 16\left(2\mu+1\right)\left(2\alpha(1+\alpha)x^2 - (1+2\alpha)x\right)\right]}.$$

4. Conclusion

Many researchers have recently worked on special functions since they are used in so many different mathematical and scientific fields. This study has successfully introduced new subfamilies of analytic functions by leveraging error functions subordinate to Gegenbauer polynomials. By establishing initial bounds for the coefficients a_2 and a_3 and deriving the Fekete-Szegő inequality, our research contributes significantly to the understanding of these functions. By applying the linear operator Ef_μ to Gegenbauer polynomials, we have gained novel insights into analytic functions within the open unit disk U. This work bridges theoretical mathematics and practical science, with applications in multiple fields. Looking ahead, potential avenues for investigation include exploring bounds for higher-order coefficients, delving into applications within physics and engineering, and extending the framework to encompass other special functions or polynomials.

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