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Research Paper

New Extension in Ostrowski's Type Inequalities by Using 13-Step Linear Kernel

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Abstract

In this paper, we present an extantion of Ostrowski's inequalities by using newly derived identity. With the help of this inequality, we build up new results for $\acute{y}' \in L_1, \acute{y}' \in L_2$, and $\acute{y}'' \in L_2$. For this purpose, our approach utilizing Grüss inequality, Diaz-Metcalf inequality and Cauchy inequality. To prove our main findings, we use a new extended kernal (13-step linear kernel), we produce some new useful results. At the end, we apply our results to numerical integration and cumulative distribution function.

Key Words: Ostrowski Inequality, Numerical Integration, 13-Step Linear Kernel

AMS 2020 Classification: 26A33, 26D07, 26D10, 26D15

1. Introduction

The research on Ostrowski type inequalities has seen significant contributions from various researchers over the years. In 1970, Metrinovic [1, 2, 3] emphasized the importance of inequalities and validated Ostrowski Type Inequalities for 2-times differentiable mappings. Subsequently, Barnet et al. [4] conducted research on Ostrowski Type Inequalities for $L_p(c, d)$ and $L_1(c, d)$. Husain et al. [5] further generalized Ostrowski Type Inequalities and presented new estimates. Qayyum et al. [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17] provided a generalized form of Ostrowski Type Inequalities for twice derivable mappings, while Dragomir and Wang [18] offered a classical method to prove Ostrowski Type Inequalities and demonstrated its applications for the first time. Barnet et al. [4] also stressed another new concept by proving Ostrowski Type Inequalities using the βfunction for 1^{st} and 2^{nd} differential mappings and applying their findings to numerical quadrature rules. Notably, development on Ostrowski type inequalities started with 2-Step kernels and later expanded to 3-step kernels (e.g., [19, 20, 21]) , and a few researchers (e.g., [9, 10]) focused on 5-step kernels.

In this paper, we extended our mapping i.e 13-step kernel.

2. Main Findings

Before to prove our main results, we have to prove the following lemma. Then with the help of this lemma, we will produce our new results.

Lemma 1. Let $\acute{y}:\left[\hat{h},\breve{a}\right]\to\mathbb{R}$ be such that $\acute{y}\,'$ is absolutely continuous on $\left[\hat{h},\breve{a}\right]$. Define the kernel $P\left(\acute{n},\dot{g}\right)$ as:

$$P(\acute{n},\dot{g})=\begin{cases}\dot{g}-\hat{h}; & \dot{g}\epsilon\left(\hat{h},\frac{31\hat{h}+\acute{n}}{32}\right]\\ \dot{g}-\frac{63\hat{h}+\breve{a}}{64}; & \dot{g}\epsilon\left(\frac{31\hat{h}+\acute{n}}{32},\frac{15\hat{h}+\acute{n}}{16}\right]\\ \dot{g}-\frac{31\hat{h}+\breve{a}}{32}; & \dot{g}\epsilon\left(\frac{15\hat{h}+\acute{n}}{16},\frac{7\hat{h}+\acute{n}}{8}\right]\\ \dot{g}-\frac{15\hat{h}+\breve{a}}{16}; & \dot{g}\epsilon\left(\frac{7\hat{h}+\acute{n}}{8},\frac{3\hat{h}+\acute{n}}{4}\right]\\ \dot{g}-\frac{7\hat{h}+\breve{a}}{8}; & \dot{g}\epsilon\left(\frac{3\hat{h}+\acute{n}}{4},\frac{\hat{h}+\acute{n}}{2}\right]\\ \dot{g}-\frac{3\hat{h}+\breve{a}}{4}; & \dot{g}\epsilon\left(\frac{\hat{h}+\acute{n}}{2},\acute{n}\right]\\ \dot{g}-\frac{\hat{h}+\breve{a}}{2}; & \dot{g}\epsilon\left(\acute{n},\hat{h}+2\breve{a}-\acute{n}\right]\\ \dot{g}-\frac{\hat{h}+3\breve{a}}{4}; & \dot{g}\epsilon\left(\hat{h}+2\breve{a}-\acute{n},\frac{\hat{h}+2\breve{a}-\acute{n}}{2}\right]\\ \dot{g}-\frac{\hat{h}+7\breve{a}}{8}; & \dot{g}\epsilon\left(\frac{\hat{h}+2\breve{a}-\acute{n}}{2},\frac{\hat{h}+4\breve{a}-\acute{n}}{4}\right]\\ \dot{g}-\frac{\hat{h}+15\breve{a}}{16}; & \dot{g}\epsilon\left(\frac{\hat{h}+4\breve{a}-\acute{n}}{4},\frac{\hat{h}+8\breve{a}-\acute{n}}{8}\right]\\ \dot{g}-\frac{\hat{h}+31\breve{a}}{32}; & \dot{g}\epsilon\left(\frac{\hat{h}+8\breve{a}-\acute{n}}{8},\frac{\hat{h}+16\breve{a}-\acute{n}}{16}\right]\\ \dot{g}-\frac{\hat{h}+63\breve{a}}{64}; & \dot{g}\epsilon\left(\frac{\hat{h}+16\breve{a}-\acute{n}}{16},\frac{\hat{h}+32\breve{a}-\acute{n}}{32}\right]\\ \dot{g}-\breve{a}; & \dot{g}\epsilon\left(\frac{\hat{h}+32\breve{a}-\acute{n}}{32},\breve{a}\right]\end{cases}\tag{1}$$

for all $\acute{n}\epsilon\left[\hat{h},\frac{\hat{h}+\breve{a}}{2}\right]$.

Proof. By integrating by parts, we have the following identity

$$\begin{aligned}\frac{1}{\breve{a}-\hat{h}}\int_{\hat{h}}^{\breve{a}}P(\acute{n},\dot{g})\acute{y}'(\dot{g})\,d\dot{g} &= \frac{1}{4}\left[\frac{1}{16}\left\{\acute{y}\left(\frac{31\hat{h}+\acute{n}}{32}\right)+\acute{y}\left(\frac{\hat{h}+32\breve{a}-\acute{n}}{32}\right)+\acute{y}\left(\frac{15\hat{h}+\acute{n}}{16}\right)\right.\right.\\ &\left.+\acute{y}\left(\frac{\hat{h}+16\breve{a}-\acute{n}}{16}\right)\right\}+\frac{1}{8}\left\{\acute{y}\left(\frac{7\hat{h}+\acute{n}}{8}\right)+\acute{y}\left(\frac{\hat{h}+8\breve{a}-\acute{n}}{8}\right)\right\}\\ &+\frac{1}{4}\left\{\acute{y}\left(\frac{3\hat{h}+\acute{n}}{4}\right)+\acute{y}\left(\frac{\hat{h}+4\breve{a}-\acute{n}}{4}\right)\right\}+\frac{1}{2}\left\{\acute{y}\left(\frac{\hat{h}+\acute{n}}{2}\right)\right.\end{aligned}\tag{2}$$

$$+\acute{y}\left(\frac{\hat{h}+2\breve{a}-\acute{n}}{2}\right)\bigg\}+\ \acute{y}(\acute{n})+\acute{y}\left(\hat{h}+\breve{a}-\acute{n}\right)\bigg]-\frac{1}{\breve{a}-\hat{h}}\int_{\hat{h}}^{\breve{a}}\acute{y}(\dot{g})\,d\dot{g}.$$

□

Now we will discuss following cases by using above lemma.

2.1. Case 1 for $\acute{y}' \epsilon L^1 \left[\hat{h}, \breve{a}\right]$

Theorem 1. *Let* $\acute{y}:\left[\hat{h},\breve{a}\right]\to\mathbb{R}$ *be differentiable on* $\left(\hat{h},\breve{a}\right)$. *If* $\acute{y}'\in L^1\left[\hat{h},\breve{a}\right]$ *is absolutely continuous on* $\left[\hat{h},\breve{a}\right]$ *and* $\gamma\le\acute{y}'(\dot{g})\le\Gamma$, *for all* $\dot{g}\in\left[\hat{h},\breve{a}\right]$, *then*

$$\Bigg|\frac{1}{4}\left[\frac{1}{16}\left\{\acute{y}\left(\frac{31\hat{h}+\acute{n}}{32}\right)+\acute{y}\left(\frac{\hat{h}+32\breve{a}-\acute{n}}{32}\right)+\acute{y}\left(\frac{15\hat{h}+\acute{n}}{16}\right)+\acute{y}\left(\frac{\hat{h}+16\breve{a}-\acute{n}}{16}\right)\right\}\right. \tag{3}$$
$$+\frac{1}{8}\left\{\acute{y}\left(\frac{7\hat{h}+\acute{n}}{8}\right)+\acute{y}\left(\frac{\hat{h}+8\breve{a}-\acute{n}}{8}\right)\right\}+\frac{1}{4}\left\{\acute{y}\left(\frac{3\hat{h}+\acute{n}}{4}\right)+\acute{y}\left(\frac{\hat{h}+4\breve{a}-\acute{n}}{4}\right)\right\}$$
$$\left.+\frac{1}{2}\left\{\acute{y}\left(\frac{\hat{h}+\acute{n}}{2}\right)+\acute{y}\left(\frac{\hat{h}+2\breve{a}-\acute{n}}{2}\right)\right\}+\ \acute{y}(\acute{n})+\acute{y}\left(\hat{h}+\breve{a}-\acute{n}\right)\right]-\frac{1}{\breve{a}-\hat{h}}\int_{\hat{h}}^{\breve{a}}\acute{y}(\dot{g})\,d\dot{g}\Bigg|$$
$$\le\frac{1}{256}\left(\breve{a}-\hat{h}\right)(\Gamma-\gamma).$$

Proof. As we know that for all $\dot{g}\in\left[\hat{h},\breve{a}\right]$ and $\acute{n}\in\left[\hat{h},\frac{\hat{h}+\breve{a}}{2}\right]$, we have

$$\acute{n}-\frac{63\hat{h}+\breve{a}}{64}\le P(\acute{n},\dot{g})\le\acute{n}-\hat{h}.$$

Apply Grüss-inequality [6] to the mappings $P(\acute{n},\dot{g})$ and $\acute{y}'(\dot{g})$, we obtain

$$\left|\frac{1}{\breve{a}-\hat{h}}\int_{\hat{h}}^{\breve{a}}P(\acute{n},\dot{g})\,\acute{y}'(\dot{g})\,d\dot{g}-\frac{1}{\left(\breve{a}-\hat{h}\right)^2}\int_{\hat{h}}^{\breve{a}}P(\acute{n},\dot{g})\,d\dot{g}\int_{\hat{h}}^{\breve{a}}\acute{y}'(\dot{g})\,d\dot{g}\right|\le\frac{1}{256}\left(\breve{a}-\hat{h}\right)(\Gamma-\gamma). \tag{4}$$

We have

$$\frac{1}{\breve{a}-\hat{h}}\int_{\hat{h}}^{\breve{a}}P(\acute{n},\dot{g})d\dot{g}=0 \tag{5}$$

and

$$\frac{1}{\breve{a}-\hat{h}}\int_{\hat{h}}^{\breve{a}}\acute{y}''(\dot{g})\,d\dot{g}=\frac{\acute{y}'(\breve{a})-\acute{y}'\left(\hat{h}\right)}{\breve{a}-\hat{h}}. \tag{6}$$

Hence using (4) − (6), we get our required result (3). □

Corollary 1. By replacing $\acute{n}=\frac{\hat{h}+\breve{a}}{2}$ in (3), we get

$$\left|\frac{1}{4}\left[\frac{1}{16}\left\{\acute{y}\left(\frac{63\hat{h}+\breve{a}}{64}\right)+\acute{y}\left(\frac{\hat{h}+63\breve{a}}{64}\right)+\acute{y}\left(\frac{31\hat{h}+\breve{a}}{32}\right)+\acute{y}\left(\frac{\hat{h}+31\breve{a}}{32}\right)\right\}\right.\right.$$

$$+\frac{1}{8}\left\{\acute{y}\left(\frac{15\hat{h}+\breve{a}}{16}\right)+\acute{y}\left(\frac{\hat{h}+15\breve{a}}{16}\right)\right\}+\frac{1}{4}\left\{\acute{y}\left(\frac{7\hat{h}+\breve{a}}{8}\right)+\acute{y}\left(\frac{\hat{h}+7\breve{a}}{8}\right)\right\}$$

$$\left.+\frac{1}{2}\left\{\acute{y}\left(\frac{3\hat{h}+\breve{a}}{4}\right)+\acute{y}\left(\frac{\hat{h}+3\breve{a}}{4}\right)\right\}+\ 2\acute{y}\left(\frac{\hat{h}+\breve{a}}{2}\right)\right]-\frac{1}{\breve{a}-\hat{h}}\int_{\hat{h}}^{\breve{a}}\acute{y}\left(\dot{g}\right)d\dot{g}\right|$$

$$\leq\frac{1}{256}\left(\breve{a}-\hat{h}\right)\left(\Gamma-\gamma\right).$$

2.2. Case 2 for $\acute{y}' \in L^1\left[\hat{h},\breve{a}\right]$

Theorem 2. *Let $I :\subset \mathbb{R} \to \mathbb{R}$ be differentiable mapping on I^0,the interior of the interval I,and let $\hat{h}, \breve{a} \in I$ with $\hat{h} < \breve{a}$.If $\acute{y}' \in L^1\left[\hat{h},\breve{a}\right]$ and $\gamma \leq \acute{y}'(\dot{g}) \leq \Gamma$, for all $\acute{n} \in \left[\hat{h},\breve{a}\right]$. Then the following inequality*

$$\left|\frac{1}{4}\left[\frac{1}{16}\left\{\acute{y}\left(\frac{31\hat{h}+\acute{n}}{32}\right)+\acute{y}\left(\frac{\hat{h}+32\breve{a}-\acute{n}}{32}\right)+\acute{y}\left(\frac{15\hat{h}+\acute{n}}{16}\right)+\acute{y}\left(\frac{\hat{h}+16\breve{a}-\acute{n}}{16}\right)\right\}\right.\right. \tag{7}$$

$$+\frac{1}{8}\left\{\acute{y}\left(\frac{7\hat{h}+\acute{n}}{8}\right)+\acute{y}\left(\frac{\hat{h}+8\breve{a}-\acute{n}}{8}\right)\right\}+\frac{1}{4}\left\{\acute{y}\left(\frac{3\hat{h}+\acute{n}}{4}\right)+\acute{y}\left(\frac{\hat{h}+4\breve{a}-\acute{n}}{4}\right)\right\}$$

$$\left.\left.+\frac{1}{2}\left\{\acute{y}\left(\frac{\hat{h}+\acute{n}}{2}\right)+\acute{y}\left(\frac{\hat{h}+2\breve{a}-\acute{n}}{2}\right)\right\}+\ \acute{y}\left(\acute{n}\right)+\acute{y}\left(\hat{h}+\breve{a}-\acute{n}\right)\right]-\frac{1}{\breve{a}-\hat{h}}\int_{\hat{h}}^{\breve{a}}\acute{y}\left(\dot{g}\right)d\dot{g}\right|$$

$$\leq\frac{1}{2048(\breve{a}-\hat{h})}\left[\left(\acute{n}-\hat{h}\right)^2+1364\left(\acute{n}-\frac{3\hat{h}+\breve{a}}{4}\right)^2-341\left(\acute{n}-\frac{\hat{h}+\breve{a}}{2}\right)^2\right]\left(\Gamma+\gamma\right).$$

holds for all $\acute{n} \in \left[\hat{h}, \frac{\hat{h}+\breve{a}}{2}\right]$.

Proof. Let $C = \frac{\Gamma+\gamma}{2}$, then

$$\frac{1}{\breve{a}-\hat{h}}\int_{\hat{h}}^{\breve{a}}P\left(\acute{n},\dot{g}\right)\acute{y}'\left(\dot{g}\right)d\dot{g}-\frac{c}{\left(\breve{a}-\hat{h}\right)}\int_{\hat{h}}^{\breve{a}}P\left(\acute{n},\dot{g}\right)d\dot{g}=\frac{1}{\breve{a}-\hat{h}}\int_{\hat{h}}^{\breve{a}}P\left(\acute{n},\dot{g}\right)\left[\acute{y}'\left(\dot{g}\right)-c\right]d\dot{g}$$

$$=\frac{1}{4}\left[\frac{1}{16}\left\{\acute{y}\left(\frac{31\hat{h}+\acute{n}}{32}\right)+\acute{y}\left(\frac{\hat{h}+32\breve{a}-\acute{n}}{32}\right)+\acute{y}\left(\frac{15\hat{h}+\acute{n}}{16}\right)+\acute{y}\left(\frac{\hat{h}+16\breve{a}-\acute{n}}{16}\right)\right\}\right.$$

$$+\frac{1}{8}\left\{\acute{y}\left(\frac{7\hat{h}+\acute{n}}{8}\right)+\acute{y}\left(\frac{\hat{h}+8\breve{a}-\acute{n}}{8}\right)\right\}+\frac{1}{4}\left\{\acute{y}\left(\frac{3\hat{h}+\acute{n}}{4}\right)+\acute{y}\left(\frac{\hat{h}+4\breve{a}-\acute{n}}{4}\right)\right\}$$

$$\left.+\frac{1}{2}\left\{\acute{y}\left(\frac{\hat{h}+\acute{n}}{2}\right)+\acute{y}\left(\frac{\hat{h}+2\breve{a}-\acute{n}}{2}\right)\right\}+\ \acute{y}\left(\acute{n}\right)+\acute{y}\left(\hat{h}+\breve{a}-\acute{n}\right)\right]-\frac{1}{\breve{a}-\hat{h}}\int_{\hat{h}}^{\breve{a}}\acute{y}\left(\dot{g}\right)d\dot{g},$$

where

$$\int_{\hat{h}}^{\breve{a}}P\left(\acute{n},\dot{g}\right)d\dot{g}=0.$$

On the other hand, we have

$$\left|\frac{1}{\breve{a}-\hat{h}}\int_{\hat{h}}^{\breve{a}} P(\acute{n},\dot{g})\left[\acute{y}'(\dot{g})-c\right]d\dot{g}\right| \le \frac{1}{\breve{a}-\hat{h}}\max_{\dot{g}\epsilon[\hat{h},\breve{a}]}\left|\acute{y}'(\dot{g})-c\right|\int_{\hat{h}}^{\breve{a}}|P(\acute{n},\dot{g})|\,d\dot{g}. \tag{8}$$

Since

$$\max_{\dot{g}\epsilon[\hat{h},\breve{a}]}\left|\acute{y}'(\dot{g})-c\right| \le \frac{\Gamma+\gamma}{2} \tag{9}$$

and

$$\frac{1}{\breve{a}-\hat{h}}\int_{\hat{h}}^{\breve{a}}|P(\acute{n},\dot{g})|\,d\dot{g} = \frac{1}{1024(\breve{a}-\hat{h})}\left[\left(\acute{n}-\hat{h}\right)^2+1364\left(\acute{n}-\frac{3\hat{h}+\breve{a}}{4}\right)^2-341\left(\acute{n}-\frac{\hat{h}+\breve{a}}{2}\right)^2\right]. \tag{10}$$

Hence using (8) − (10), we get our required result (7). □

Corollary 2. By replacing $\acute{n}=\frac{\hat{h}+\breve{a}}{2}$ in (7), we get

$$\begin{aligned}&\left|\frac{1}{4}\left[\frac{1}{16}\left\{\acute{y}\left(\frac{63\hat{h}+\breve{a}}{64}\right)+\acute{y}\left(\frac{\hat{h}+63\breve{a}}{64}\right)+\acute{y}\left(\frac{31\hat{h}+\breve{a}}{32}\right)+\acute{y}\left(\frac{\hat{h}+31\breve{a}}{32}\right)\right\}\right.\right.\\&+\frac{1}{8}\left\{\acute{y}\left(\frac{15\hat{h}+\breve{a}}{16}\right)+\acute{y}\left(\frac{\hat{h}+15\breve{a}}{16}\right)\right\}+\frac{1}{4}\left\{\acute{y}\left(\frac{7\hat{h}+\breve{a}}{8}\right)+\acute{y}\left(\frac{\hat{h}+7\breve{a}}{8}\right)\right\}\\&\left.\left.+\frac{1}{2}\left\{\acute{y}\left(\frac{3\hat{h}+\breve{a}}{4}\right)+\acute{y}\left(\frac{\hat{h}+3\breve{a}}{4}\right)\right\}+\ 2\acute{y}\left(\frac{\hat{h}+\breve{a}}{2}\right)\right]-\frac{1}{\breve{a}-\hat{h}}\int_{\hat{h}}^{\breve{a}}\acute{y}(\dot{g})\,d\dot{g}\right|\\&\le\frac{171}{4096}\left(\breve{a}-\hat{h}\right)(\Gamma+\gamma).\end{aligned}$$

2.3. Case 3 for $\acute{y}' \in L^1\left[\hat{h},\breve{a}\right]$

Theorem 3. *Let* $\acute{y}:\left[\hat{h},\breve{a}\right]\to\mathbb{R}$ *be differentiable mapping on* $\left(\hat{h},\breve{a}\right)$. *If* $\acute{y}'\in L^1\left[\hat{h},\breve{a}\right]$ *and* $\gamma\le\acute{y}'(\dot{g})\le\Gamma$, *for all* $\dot{g}\in\left[\hat{h},\breve{a}\right]$, *then*

$$\begin{aligned}&\left|\frac{1}{4}\left[\frac{1}{16}\left\{\acute{y}\left(\frac{31\hat{h}+\acute{n}}{32}\right)+\acute{y}\left(\frac{\hat{h}+32\breve{a}-\acute{n}}{32}\right)+\acute{y}\left(\frac{15\hat{h}+\acute{n}}{16}\right)+\acute{y}\left(\frac{\hat{h}+16\breve{a}-\acute{n}}{16}\right)\right.\right.\right.\\&+\frac{1}{8}\left\{\acute{y}\left(\frac{7\hat{h}+\acute{n}}{8}\right)+\acute{y}\left(\frac{\hat{h}+8\breve{a}-\acute{n}}{8}\right)\right\}+\frac{1}{4}\left\{\acute{y}\left(\frac{3\hat{h}+\acute{n}}{4}\right)+\acute{y}\left(\frac{\hat{h}+4\breve{a}-\acute{n}}{4}\right)\right\}\\&\left.\left.+\frac{1}{2}\left\{\acute{y}\left(\frac{\hat{h}+\acute{n}}{2}\right)+\acute{y}\left(\frac{\hat{h}+2\breve{a}-\acute{n}}{2}\right)\right\}+\ \acute{y}(\acute{n})+\acute{y}\left(\hat{h}+\breve{a}-\acute{n}\right)\right]-\frac{1}{\breve{a}-\hat{h}}\int_{\hat{h}}^{\breve{a}}\acute{y}(\dot{g})\,d\dot{g}\right|\\&\le\Omega(S-\gamma),\end{aligned} \tag{11}$$

and

$$\left|\frac{1}{4}\left[\frac{1}{16}\left\{\acute{y}\left(\frac{31\hat{h}+\acute{n}}{32}\right)+\acute{y}\left(\frac{\hat{h}+32\breve{a}-\acute{n}}{32}\right)+\acute{y}\left(\frac{15\hat{h}+\acute{n}}{16}\right)+\acute{y}\left(\frac{\hat{h}+16\breve{a}-\acute{n}}{16}\right)\right.\right.\right. \tag{12}$$

$$+\frac{1}{8}\left\{\acute{y}\left(\frac{7\hat{h}+\acute{n}}{8}\right)+\acute{y}\left(\frac{\hat{h}+8\breve{a}-\acute{n}}{8}\right)\right\}+\frac{1}{4}\left\{\acute{y}\left(\frac{3\hat{h}+\acute{n}}{4}\right)+\acute{y}\left(\frac{\hat{h}+4\breve{a}-\acute{n}}{4}\right)\right\}$$

$$+\frac{1}{2}\left\{\acute{y}\left(\frac{\hat{h}+\acute{n}}{2}\right)+\acute{y}\left(\frac{\hat{h}+2\breve{a}-\acute{n}}{2}\right)\right\}+\ \acute{y}\left(\acute{n}\right)+\acute{y}\left(\hat{h}+\breve{a}-\acute{n}\right)\Bigg]-\frac{1}{\breve{a}-\hat{h}}\int\limits_{\hat{h}}^{\breve{a}}\acute{y}\left(\dot{g}\right)d\dot{g}\Bigg|$$

$$\leq\Omega\left(S-\Gamma\right),$$

for all $\acute{n}\in\left[\hat{h},\frac{\hat{h}+\breve{a}}{2}\right]$, *where*

$$\Omega = \max_{\dot{g}\epsilon\left[\hat{h},\breve{a}\right]}\left|p(\acute{n},\dot{g})\right|,$$

$$S = \frac{\acute{y}(\breve{a})-\acute{y}(\hat{h})}{\breve{a}-\hat{h}},$$

$$\gamma = \inf_{\dot{g}\epsilon\left[\hat{h},\breve{a}\right]}\acute{y}^{'}\left(\dot{g}\right),$$

$$\Gamma = \sup_{\dot{g}\epsilon\left[\hat{h},\breve{a}\right]}\acute{y}^{'}\left(\dot{g}\right).$$

Proof. We know that

$$\frac{1}{\breve{a}-\hat{h}}\int\limits_{\hat{h}}^{\breve{a}}P\left(\acute{n},\dot{g}\right)\acute{y}^{'}\left(\dot{g}\right)d\dot{g}-\frac{1}{\left(\breve{a}-\hat{h}\right)^{2}}\int\limits_{\hat{h}}^{\breve{a}}P\left(\acute{n},\dot{g}\right)d\dot{g}\int\limits_{\hat{h}}^{\breve{a}}\acute{y}^{'}\left(\dot{g}\right)d\dot{g} \tag{13}$$

$$=\frac{1}{4}\left[\frac{1}{16}\left\{\acute{y}\left(\frac{31\hat{h}+\acute{n}}{32}\right)+\acute{y}\left(\frac{\hat{h}+32\breve{a}-\acute{n}}{32}\right)+\acute{y}\left(\frac{15\hat{h}+\acute{n}}{16}\right)\right.\right.$$

$$\left.+\acute{y}\left(\frac{\hat{h}+16\breve{a}-\acute{n}}{16}\right)\right\}+\frac{1}{8}\left\{\acute{y}\left(\frac{7\hat{h}+\acute{n}}{8}\right)+\acute{y}\left(\frac{\hat{h}+8\breve{a}-\acute{n}}{8}\right)\right\}$$

$$+\frac{1}{4}\left\{\acute{y}\left(\frac{3\hat{h}+\acute{n}}{4}\right)+\acute{y}\left(\frac{\hat{h}+4\breve{a}-\acute{n}}{4}\right)\right\}+\frac{1}{2}\left\{\acute{y}\left(\frac{\hat{h}+\acute{n}}{2}\right)\right.$$

$$\left.\left.+\acute{y}\left(\frac{\hat{h}+2\breve{a}-\acute{n}}{2}\right)\right\}+\ \acute{y}\left(\acute{n}\right)+\acute{y}\left(\hat{h}+\breve{a}-\acute{n}\right)\right]-\frac{1}{\breve{a}-\hat{h}}\int\limits_{\hat{h}}^{\breve{a}}\acute{y}\left(\dot{g}\right)d\dot{g}.$$

We denote

$$R_{n}(\acute{n})=\frac{1}{\breve{a}-\hat{h}}\int\limits_{\hat{h}}^{\breve{a}}P\left(\acute{n},\dot{g}\right)\acute{y}^{'}\left(\dot{g}\right)d\dot{g}-\frac{1}{\left(\breve{a}-\hat{h}\right)^{2}}\int\limits_{\hat{h}}^{\breve{a}}P\left(\acute{n},\dot{g}\right)d\dot{g}\int\limits_{\hat{h}}^{\breve{a}}\acute{y}^{'}\left(\dot{g}\right)d\dot{g}. \tag{14}$$

If $C\in\mathbb{R}$ is an arbitrary constant, then we have

$$R_{n}(\acute{n})=\frac{1}{\breve{a}-\hat{h}}\int\limits_{\hat{h}}^{\breve{a}}\left(\acute{y}^{'}\left(\dot{g}\right)-C\right)\left[P\left(\acute{n},\dot{g}\right)-\frac{1}{\breve{a}-\hat{h}}\int\limits_{\hat{h}}^{\breve{a}}P\left(\acute{n},s\right)ds\right]d\dot{g}. \tag{15}$$

Since

$$\int\limits_{\hat{h}}^{\breve{a}}\left[P\left(\acute{n},\dot{g}\right)-\frac{1}{\breve{a}-\hat{h}}\int\limits_{\hat{h}}^{\breve{a}}P\left(\acute{n},s\right)ds\right]d\dot{g}=0.$$

Furthermore, we have

$$|R_n(\acute{n})| \leq \frac{1}{\breve{a}-\hat{h}} \max_{\dot{g}\epsilon[\hat{h},\breve{a}]} |p(\acute{n},\dot{g})-0| \int_{\hat{h}}^{\breve{a}} \left|(\acute{y}'(\dot{g})-C)\right| d\dot{g} = 0,$$

$$\max_{\dot{g}\epsilon[\hat{h},\breve{a}]} |p(\acute{n},\dot{g})| = \Omega. \tag{16}$$

We also have

$$\int_{\hat{h}}^{\breve{a}} \left|(\acute{y}'(\dot{g})-\gamma)\right| d\dot{g} = (S-\gamma)(\breve{a}-\hat{h}), \tag{17}$$

$$\int_{\hat{h}}^{\breve{a}} \left|(\acute{y}'(\dot{g})-\Gamma\right| d\dot{g} = (\Gamma-S)(\breve{a}-\hat{h}). \tag{18}$$

Hence using(6) and (13) − (18) , we obtain (11) and (12) . □

2.4. Case 4 for $\acute{y}' \in L^2\left[\hat{h},\breve{a}\right]$

Theorem 4. *Let $\acute{y}:\left[\hat{h},\breve{a}\right]\to\mathbb{R}$ be three times differentiable function on $\left(\hat{h},\breve{a}\right)$. If $\acute{y}'\in L^2\left[\hat{h},\breve{a}\right]$, then for all $\acute{n}\in\left[\hat{h},\frac{\hat{h}+\breve{a}}{2}\right]$, we have*

$$\left|\frac{1}{4}\left[\frac{1}{16}\left\{\acute{y}\left(\frac{31\hat{h}+\acute{n}}{32}\right)+\acute{y}\left(\frac{\hat{h}+32\breve{a}-\acute{n}}{32}\right)+\acute{y}\left(\frac{15\hat{h}+\acute{n}}{16}\right)+\acute{y}\left(\frac{\hat{h}+16\breve{a}-\acute{n}}{16}\right)\right\}\right.\right. \tag{19}$$
$$+\frac{1}{8}\left\{\acute{y}\left(\frac{7\hat{h}+\acute{n}}{8}\right)+\acute{y}\left(\frac{\hat{h}+8\breve{a}-\acute{n}}{8}\right)\right\}+\frac{1}{4}\left\{\acute{y}\left(\frac{3\hat{h}+\acute{n}}{4}\right)+\acute{y}\left(\frac{\hat{h}+4\breve{a}-\acute{n}}{4}\right)\right\}$$
$$\left.\left.+\frac{1}{2}\left\{\acute{y}\left(\frac{\hat{h}+\acute{n}}{2}\right)+\acute{y}\left(\frac{\hat{h}+2\breve{a}-\acute{n}}{2}\right)\right\}+\acute{y}(\acute{n})+\acute{y}\left(\hat{h}+\breve{a}-\acute{n}\right)\right]-\frac{1}{\breve{a}-\hat{h}}\int_{\hat{h}}^{\breve{a}}\acute{y}(\dot{g})\,d\dot{g}\right|$$
$$\leq\sqrt{\frac{\sigma(\acute{y}')}{\breve{a}-\hat{h}}}\left[\frac{1}{49152}\left(\acute{n}-\hat{h}\right)^2+\frac{4681}{6144}\left(\acute{n}-\frac{3\hat{h}+\breve{a}}{4}\right)^2-\frac{12483}{16384}\left(\acute{n}-\frac{\hat{h}+\breve{a}}{2}\right)^2\right],$$

where

$$\sigma(\acute{y}') = ||\acute{y}'''||_2^2 - \frac{\acute{y}'(\breve{a})-\acute{y}'\left(\hat{h}\right)}{\breve{a}-\hat{h}} = ||\acute{y}'''||_2^2 - S^2\left(\breve{a}-\hat{h}\right).$$

Proof. Let $R_n(\acute{n})$ be defined as in (14) then from (13) , we get

$$R_n(\acute{n}) = \left|\frac{1}{4}\left[\frac{1}{16}\left\{\acute{y}\left(\frac{31\hat{h}+\acute{n}}{32}\right)+\acute{y}\left(\frac{\hat{h}+32\breve{a}-\acute{n}}{32}\right)+\acute{y}\left(\frac{15\hat{h}+\acute{n}}{16}\right)+\acute{y}\left(\frac{\hat{h}+16\breve{a}-\acute{n}}{16}\right)\right\}\right.\right.$$
$$+\frac{1}{8}\left\{\acute{y}\left(\frac{7\hat{h}+\acute{n}}{8}\right)+\acute{y}\left(\frac{\hat{h}+8\breve{a}-\acute{n}}{8}\right)\right\}+\frac{1}{4}\left\{\acute{y}\left(\frac{3\hat{h}+\acute{n}}{4}\right)+\acute{y}\left(\frac{\hat{h}+4\breve{a}-\acute{n}}{4}\right)\right\}$$

$$+\frac{1}{2}\left\{\acute{y}\left(\frac{\hat{h}+\acute{n}}{2}\right)+\acute{y}\left(\frac{\hat{h}+2\breve{a}-\acute{n}}{2}\right)\right\}+\ \acute{y}\left(\acute{n}\right)+\acute{y}\left(\hat{h}+\breve{a}-\acute{n}\right)\Bigg]-\frac{1}{\breve{a}-\hat{h}}\int_{\hat{h}}^{\breve{a}}\acute{y}\left(\dot{g}\right)d\dot{g}.\Bigg|$$

If we choose $C=\frac{1}{\breve{a}-\hat{h}}\int_{\hat{h}}^{\breve{a}}\acute{y}^{'}\left(s\right)ds$ in (15) and the Cauchy inequality, we get

$$\left|R_n\left(\acute{n}\right)\right|\leq\frac{1}{\breve{a}-\hat{h}}\int_{\hat{h}}^{\breve{a}}\left|\acute{y}^{'}\left(\dot{g}\right)-\frac{1}{\breve{a}-\hat{h}}\int_{\hat{h}}^{\breve{a}}\acute{y}^{'}\left(s\right)ds\right|\left|P\left(\acute{n},\dot{g}\right)-\frac{1}{\breve{a}-\hat{h}}\int_{\hat{h}}^{\breve{a}}P\left(\acute{n},s\right)ds\right|d\dot{g}$$

$$\leq\frac{1}{\breve{a}-\hat{h}}\left[\int_{\hat{h}}^{\breve{a}}\left(\acute{y}^{'}\left(\dot{g}\right)-\frac{1}{\breve{a}-\hat{h}}\int_{\hat{h}}^{\breve{a}}\acute{y}^{'}\left(s\right)ds\right)^{2}d\dot{g}\right]^{\frac{1}{2}}\left[\int_{\hat{h}}^{\breve{a}}\left(P\left(\acute{n},\dot{g}\right)-\frac{1}{\breve{a}-\hat{h}}\int_{\hat{h}}^{\breve{a}}P\left(\acute{n},s\right)ds\right)^{2}d\dot{g}\right]^{\frac{1}{2}}$$

and

$$\int_{\hat{h}}^{\breve{a}}\left(P\left(\acute{n},\dot{g}\right)-\frac{1}{\breve{a}-\hat{h}}\int_{\hat{h}}^{\breve{a}}P\left(\acute{n},s\right)ds\right)^{2}d\dot{g}=\int_{\hat{h}}^{\breve{a}}P\left(\acute{n},\dot{g}\right)^{2}d\dot{g}.$$

Where

$$\int_{\hat{h}}^{\breve{a}}P(\acute{n},\dot{g})^{2}d\dot{g}=\frac{1}{49152}\left(\acute{n}-\hat{h}\right)^{3}+\frac{9362}{12288}\left(\acute{n}-\frac{3\hat{h}+\breve{a}}{4}\right)^{3}-\frac{74898}{98304}\left(\acute{n}-\frac{\hat{h}+\breve{a}}{2}\right)^{3}.$$

By using above equations we get (12). ☐

Corollary 3. By replacing $\acute{n}=\frac{\hat{h}+\breve{a}}{2}$ in (19), then

$$\Bigg|\frac{1}{4}\left[\frac{1}{16}\left\{\acute{y}\left(\frac{63\hat{h}+\breve{a}}{64}\right)+\acute{y}\left(\frac{\hat{h}+63\breve{a}}{64}\right)+\acute{y}\left(\frac{31\hat{h}+\breve{a}}{32}\right)+\acute{y}\left(\frac{\hat{h}+31\breve{a}}{32}\right)\right\}\right.$$
$$+\frac{1}{8}\left\{\acute{y}\left(\frac{15\hat{h}+\breve{a}}{16}\right)+\acute{y}\left(\frac{\hat{h}+15\breve{a}}{16}\right)\right\}+\frac{1}{4}\left\{\acute{y}\left(\frac{7\hat{h}+\breve{a}}{8}\right)+\acute{y}\left(\frac{\hat{h}+7\breve{a}}{8}\right)\right\}$$
$$\left.+\frac{1}{2}\left\{\acute{y}\left(\frac{3\hat{h}+\breve{a}}{4}\right)+\acute{y}\left(\frac{\hat{h}+3\breve{a}}{4}\right)\right\}+\ 2\acute{y}\left(\frac{\hat{h}+\breve{a}}{2}\right)\right]-\frac{1}{\breve{a}-\hat{h}}\int_{\hat{h}}^{\breve{a}}\acute{y}\left(\dot{g}\right)d\dot{g}\Bigg|$$
$$\leq\sqrt{\sigma\left(\acute{y}'\right)}\left(\breve{a}-\hat{h}\right)\frac{\sqrt{7023}}{768}.$$

2.5. Case 5 for $\acute{y}'' \in L^2\left[\hat{h},\breve{a}\right]$

Theorem 5. *Let* $\acute{y}:\left[\hat{h},\breve{a}\right]\to\mathbb{R}$ *be a twice absolutely continuous differentiable mapping on* $\left(\hat{h},\breve{a}\right)$, *with* $\acute{y}''\in L^2\left[\hat{h},\breve{a}\right]$, *then*

$$\Bigg|\frac{1}{4}\left[\frac{1}{16}\left\{\acute{y}\left(\frac{31\hat{h}+\acute{n}}{32}\right)+\acute{y}\left(\frac{\hat{h}+32\breve{a}-\acute{n}}{32}\right)+\acute{y}\left(\frac{15\hat{h}+\acute{n}}{16}\right)+\acute{y}\left(\frac{\hat{h}+16\breve{a}-\acute{n}}{16}\right)\right\}\right. \tag{20}$$
$$+\frac{1}{8}\left\{\acute{y}\left(\frac{7\hat{h}+\acute{n}}{8}\right)+\acute{y}\left(\frac{\hat{h}+8\breve{a}-\acute{n}}{8}\right)\right\}+\frac{1}{4}\left\{\acute{y}\left(\frac{3\hat{h}+\acute{n}}{4}\right)+\acute{y}\left(\frac{\hat{h}+4\breve{a}-\acute{n}}{4}\right)\right\}$$

$$+\frac{1}{2}\left\{\acute{y}\left(\frac{\hat{h}+\acute{n}}{2}\right)+\acute{y}\left(\frac{\hat{h}+2\breve{a}-\acute{n}}{2}\right)\right\}+\ \acute{y}\left(\acute{n}\right)+\acute{y}\left(\hat{h}+\breve{a}-\acute{n}\right)\Bigg]-\frac{1}{\breve{a}-\hat{h}}\int_{\hat{h}}^{\breve{a}}\acute{y}\left(\dot{g}\right)d\dot{g}\Bigg|$$

$$\leq\frac{1}{\pi}\left|\left|\acute{y}'''\right|\right|_{2}\left[\frac{1}{49152}\left(\acute{n}-\hat{h}\right)^{2}+\frac{4681}{6144}\left(\acute{n}-\frac{3\hat{h}+\breve{a}}{4}\right)^{2}-\frac{12483}{16384}\left(\acute{n}-\frac{\hat{h}+\breve{a}}{2}\right)^{2}\right].$$

for all $\acute{n}\in\left[\hat{h},\frac{\hat{h}+\breve{a}}{2}\right]$.

Proof. Let $R_n\left(\acute{n}\right)$ be used in (14) then by (13), we have

$$R_n\left(\acute{n}\right)=\Bigg|\frac{1}{4}\Bigg[\frac{1}{16}\left\{\acute{y}\left(\frac{31\hat{h}+\acute{n}}{32}\right)+\acute{y}\left(\frac{\hat{h}+32\breve{a}-\acute{n}}{32}\right)+\acute{y}\left(\frac{15\hat{h}+\acute{n}}{16}\right)\right.$$

$$\left.+\acute{y}\left(\frac{\hat{h}+16\breve{a}-\acute{n}}{16}\right)\right\}+\frac{1}{8}\left\{\acute{y}\left(\frac{7\hat{h}+\acute{n}}{8}\right)+\acute{y}\left(\frac{\hat{h}+8\breve{a}-\acute{n}}{8}\right)\right\}$$

$$+\frac{1}{4}\left\{\acute{y}\left(\frac{3\hat{h}+\acute{n}}{4}\right)+\acute{y}\left(\frac{\hat{h}+4\breve{a}-\acute{n}}{4}\right)\right\}+\frac{1}{2}\left\{\acute{y}\left(\frac{\hat{h}+\acute{n}}{2}\right)+\acute{y}\left(\frac{\hat{h}+2\breve{a}-\acute{n}}{2}\right)\right\}$$

$$+\ \acute{y}\left(\acute{n}\right)+\acute{y}\left(\hat{h}+\breve{a}-\acute{n}\right)\Bigg]-\frac{1}{\breve{a}-\hat{h}}\int_{\hat{h}}^{\breve{a}}\acute{y}\left(\dot{g}\right)d\dot{g}\Bigg|.$$

If we choose

$$C=\acute{y}'\left(\frac{\hat{h}+\breve{a}}{2}\right)$$

in (15) and also use the cauchy inequality, we have

$$\left|R_n\left(\acute{n}\right)\right|\leq\frac{1}{\breve{a}-\hat{h}}\int_{\hat{h}}^{\breve{a}}\left|\acute{y}'\left(\dot{g}\right)-\acute{y}'\left(\frac{\hat{h}+\breve{a}}{2}\right)\right|\left|P\left(\acute{n},\dot{g}\right)-\frac{1}{\breve{a}-\hat{h}}\int_{\hat{h}}^{\breve{a}}P\left(\acute{n},s\right)ds\right|d\dot{g}.$$

$$\frac{1}{\breve{a}-\hat{h}}\left[\int_{\hat{h}}^{\breve{a}}\left(\acute{y}'\left(\dot{g}\right)-\acute{y}'\left(\frac{\hat{h}+\breve{a}}{2}\right)\right)^{2}d\dot{g}\right]^{\frac{1}{2}}\left[\int_{\hat{h}}^{\breve{a}}\left(P\left(\acute{n},\dot{g}\right)-\frac{1}{\breve{a}-\hat{h}}\int_{\hat{h}}^{\breve{a}}P\left(\acute{n},s\right)ds\right)^{2}d\dot{g}\right]^{\frac{1}{2}}.$$

By using the Diaz-Metcalf Inequality to obtain

$$\int_{\hat{h}}^{\breve{a}}\left(\acute{y}'\left(\dot{g}\right)-\acute{y}'\left(\frac{\hat{h}+\breve{a}}{2}\right)\right)^{2}d\dot{g}\leq\frac{\left(\breve{a}-\hat{h}\right)^{2}}{\pi}\left|\left|\acute{y}'''\right|\right|_{2}^{2},$$

we also have

$$\int_{\hat{h}}^{\breve{a}}\left(P\left(\acute{n},\dot{g}\right)-\frac{1}{\breve{a}-\hat{h}}\int_{\hat{h}}^{\breve{a}}P\left(\acute{n},s\right)ds\right)^{2}d\dot{g}=\int_{\hat{h}}^{\breve{a}}P(\acute{n},\dot{g})^{2}d\dot{g}$$

$$=\frac{1}{49152}\left(\acute{n}-\hat{h}\right)^{3}+\frac{9362}{12288}\left(\acute{n}-\frac{3\hat{h}+\breve{a}}{4}\right)^{3}-\frac{74898}{98304}\left(\acute{n}-\frac{\hat{h}+\breve{a}}{2}\right)^{3}.$$

Hence, we proved our required result (20). □

Corollary 4. By replacing $\acute{n} = \frac{\hat{h}+\breve{a}}{2}$ in (20), then

$$\left| \frac{1}{4}\left[\frac{1}{16}\left\{ \acute{y}\left(\frac{63\hat{h}+\breve{a}}{64}\right) + \acute{y}\left(\frac{\hat{h}+63\breve{a}}{64}\right) + \acute{y}\left(\frac{31\hat{h}+\breve{a}}{32}\right) + \acute{y}\left(\frac{\hat{h}+31\breve{a}}{32}\right) \right\} \right.\right.$$
$$+\frac{1}{8}\left\{ \acute{y}\left(\frac{15\hat{h}+\breve{a}}{16}\right) + \acute{y}\left(\frac{\hat{h}+15\breve{a}}{16}\right) \right\} + \frac{1}{4}\left\{ \acute{y}\left(\frac{7\hat{h}+\breve{a}}{8}\right) + \acute{y}\left(\frac{\hat{h}+7\breve{a}}{8}\right) \right\}$$
$$\left.\left. +\frac{1}{2}\left\{ \acute{y}\left(\frac{3\hat{h}+\breve{a}}{4}\right) + \acute{y}\left(\frac{\hat{h}+3\breve{a}}{4}\right) \right\} + \ 2\acute{y}\left(\frac{\hat{h}+\breve{a}}{2}\right) \right] - \frac{1}{\breve{a}-\hat{h}} \int_{\hat{h}}^{\breve{a}} \acute{y}(\dot{g})\, d\dot{g} \right|$$
$$\leq \frac{1}{\pi} \left\| \acute{y}''' \right\|_2 \left(\breve{a}-\hat{h}\right)^{\frac{3}{2}} \frac{\sqrt{7023}}{768}.$$

3. An Application to Cumulative Distribution Function

Let X be a random variable taking values in the finite interval $\left[\hat{h}, \breve{a}\right]$ with the probability density function $\acute{y} : \left[\hat{h}, \breve{a}\right] \to [0,1]$ and cumulative distributive function

$$F(\acute{n}) = \Pr(X \leq \acute{n}) = \int_{\hat{h}}^{\acute{n}} \acute{y}(\dot{g})\, d\dot{g}, \tag{21}$$

$$F(\breve{a}) = \Pr(X \leq \breve{a}) = \int_{\hat{h}}^{\breve{a}} \acute{y}(u) du = 1. \tag{22}$$

Theorem 6. *With the assumption of Theorem 1 we have the following inequality which holds*

$$\left| \frac{\breve{a}-E(X)}{\breve{a}-\hat{h}} - \frac{1}{4}\left[\frac{1}{16}\left\{ F\left(\frac{31\hat{h}+\acute{n}}{32}\right) + F\left(\frac{\hat{h}+32\breve{a}-\acute{n}}{32}\right) + F\left(\frac{15\hat{h}+\acute{n}}{16}\right) \right.\right.\right. \tag{23}$$
$$\left. +F\left(\frac{\hat{h}+16\breve{a}-\acute{n}}{16}\right) \right\} + \frac{1}{8}\left\{ F\left(\frac{7\hat{h}+\acute{n}}{8}\right) + F\left(\frac{\hat{h}+8\breve{a}-\acute{n}}{8}\right) \right\} + \frac{1}{4}\left\{ F\left(\frac{3\hat{h}+\acute{n}}{4}\right) \right.$$
$$\left.\left.\left. +F\left(\frac{\hat{h}+4\breve{a}-\acute{n}}{4}\right) \right\} + \frac{1}{2}\left\{ F\left(\frac{\hat{h}+\acute{n}}{2}\right) + F\left(\frac{\hat{h}+2\breve{a}-\acute{n}}{2}\right) \right\} + \ F(\acute{n}) + F\left(\hat{h}+\breve{a}-\acute{n}\right) \right] \right|$$
$$\leq \frac{1}{256}\left(\breve{a}-\hat{h}\right)(\Gamma-\gamma).$$

for all $\acute{n} \in \left[\hat{h}, \frac{\hat{h}+\breve{a}}{2}\right]$, *where* $E(X)$ *is the expectation of* X.

Proof. In the proof of Theorem 1 let $f = F$ and using the fact that

$$E(X) = \int_{\hat{h}}^{\breve{a}} \dot{g} dF(\dot{g}) = b - \int_{\hat{h}}^{\breve{a}} F(\dot{g})\, d\dot{g}.$$

□

Corollary 5. By replacing $\acute{n}=\frac{\hat{h}+\breve{a}}{2}$ in (21), then

$$\left|\frac{\breve{a}-E(X)}{\breve{a}-\hat{h}}-\frac{1}{4}\left[\frac{1}{16}\left\{F\left(\frac{63\hat{h}+\breve{a}}{64}\right)+F\left(\frac{\hat{h}+63\breve{a}}{64}\right)+F\left(\frac{31\hat{h}+\breve{a}}{32}\right)\right.\right.\right.$$
$$\left.+F\left(\frac{\hat{h}+31\breve{a}}{32}\right)\right\}+\frac{1}{8}\left\{F\left(\frac{15\hat{h}+\breve{a}}{16}\right)+F\left(\frac{\hat{h}+15\breve{a}}{16}\right)\right\}+\frac{1}{4}\left\{F\left(\frac{7\hat{h}+\breve{a}}{8}\right)\right.$$
$$\left.\left.\left.+F\left(\frac{\hat{h}+7\breve{a}}{8}\right)\right\}+\frac{1}{2}\left\{F\left(\frac{3\hat{h}+\breve{a}}{4}\right)+F\left(\frac{\hat{h}+3\breve{a}}{4}\right)\right\}+\ 2F\left(\frac{\hat{h}+\breve{a}}{2}\right)\right]\right|$$
$$\leq\frac{1}{256}\left(\breve{a}-\hat{h}\right)(\Gamma-\gamma).$$

4. Conclusion

In this paper, we constructed the new extension version of Ostrowski's type inequalities for various norms by using well known inequalities. Furthermore, we also discussed some perturbed results. Notably, we developed a new peano kernel i.e. 13-step linear kernel. Finally, we applied our results for cumulative distribution function. In future, anyone can extend our results for n-times differentiable mappings even for the function of bounded variation.

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