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Research Paper

On the Well-Defined Solutions of a Nonlinear Difference Equation with Variable Coefficients

Raafat Abo-Zeid,[1,*] Mehmet Gümüş,[2,†] Ana Catarina Carapito[3,‡] and Abdul Qadeer Khan[4,§]

[1]Department of Basic Science, The Higher Institute for Engineering & Technology, Al-Obour, Cairo, Egypt,

[2]Department of Mathematics, Faculty of Science, Zonguldak Bülent Ecevit University, Zonguldak, Türkiye,

[3]Center of Mathematics and Applications of University of Beira Interior (CMA-UBI), Department of Mathematics, University of Beira Interior, Covilha, Portugal,

[4]Department of Mathematics, University of Azad Jammu and Kashmir, Muzaffarabad, 13100, Pakistan,

[*]To whom correspondence should be addressed: abuzead73@yahoo.com

[†]m.gumus@beun.edu.tr [‡]ana.carapito@ubi.pt [§]abdulqadeerkhan1@gmail.com

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Abstract

Obtaining analytical solutions of nonlinear difference equations is of great importance for a full understanding of the long-term behavior of dynamical systems. Such equations are widely used in mathematical modeling of complex systems, especially population dynamics, the spread of infectious diseases, economic models, and biological processes. Therefore, obtaining analytical solutions of nonlinear difference equations is indispensable not only for theoretical purposes but also for making reliable predictions in applied sciences. Based on this undeniable fact, in this paper, we study the following non-linear difference equation

$$\xi_{n+1} = \frac{\xi_n \xi_{n-1}}{a_n \xi_n - b_n \xi_{n-2}}, \; n \in \mathbb{N}_0,$$

where $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$ are periodic sequences of positive real numbers with prime period two and with real initial values $\xi_{-2}, \xi_{-1}, \xi_0$. We introduce the well-defined solutions and study their global behavior.

Key Words: Difference equation; Unbounded solution; Periodic solution; Forbidden set

AMS 2020 Classification: 39A10; 39A20; 40A05

1. Introduction

Difference equations have very useful applications in many fields, due to their nature (See [1, 2, 3, 4]). Difference equations that contain a quadratic term is an important type of difference equations due to the rich behavior of their solutions. For difference equations of second order with quadratic term (See [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]).

In [19], we discussed the behavior of the solutions of the difference equation

$$x_{n+1} = \frac{a x_n x_{n-1}}{b x_n - c x_{n-2}}, \ n \in \mathbb{N}_0, \tag{1}$$

where $a,\ b,\ c \in (0, \infty)$ and the initial conditions x_{-2}, x_{-1}, x_0 are real numbers.

Motivated by Equation (1), we introduce and investigate the behavior of the well-defined solutions of the difference equation (of quadratic term)

$$\xi_{n+1} = \frac{\xi_n \xi_{n-1}}{a_n \xi_n - b_n \xi_{n-2}}, \ n \in \mathbb{N}_0, \tag{2}$$

where $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$ are periodic sequences of positive real numbers with prime period two and the initial values $\xi_{-2}, \xi_{-1}, \xi_0$ are real numbers.

For more results on difference equations with quadratic term (and n^{th} degree term) of third and higher order (See[20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33] and the references therein).

2. Well-Defined Solutions of Equation (2)

This section is devoted to providing the explicit formula for the well-defined solutions of the difference equation (2).

We assume in this section that

$$a_0 \neq b_0 a_1 \text{ and } a_1 \neq b_1 a_0.$$

Theorem 1. *Let $\{\xi_n\}_{n=-2}^{\infty}$ be a well-defined solution of Equation (2). If $b_0 b_1 \neq 1$, then*

$$\xi_n = \begin{cases} \xi_{-1} \prod\limits_{j=0}^{\frac{n-1}{2}} \dfrac{1 - b_0 b_1}{(b_0 b_1)^j \mu - b_0 a_1 + a_0} & , n = 1, 3, 5, ..., \\ \xi_0 \prod\limits_{j=0}^{\frac{n-2}{2}} \dfrac{1 - b_0 b_1}{(b_0 b_1)^{j+1} \nu - b_1 a_0 + a_1} & , n = 2, 4, 6, ..., \end{cases} \tag{3}$$

where $\nu = \frac{1}{\alpha}\left(1 - b_0 b_1 - \alpha(-b_1 a_0 + a_1)\right)$, $\mu = \frac{1}{\alpha}\left(-b_0(1 - b_0 b_1) + \alpha b_0(-b_1 a_0 + a_1)\right)$ and $\alpha = \frac{\xi_0}{\xi_{-2}}$.

Proof. Solution (3) can be written as

$$\xi_{2n+i} = \xi_{-2+i} \prod_{j=0}^{n} \psi_i(j), \ i = 1, 2 \text{ and } n = 0, 1, ..., \tag{4}$$

where

$$\psi_1(j) = \frac{1 - b_0 b_1}{(b_0 b_1)^j \mu - b_0 a_1 + a_0} \text{ and } \psi_2(j) = \frac{1 - b_0 b_1}{(b_0 b_1)^{j+1} \nu - b_1 a_0 + a_1},$$

such that

$$\nu = \frac{1}{\alpha}(1 - b_0 b_1 - \alpha(-b_1 a_0 + a_1)), \ \mu = \frac{1}{\alpha}(-b_0(1 - b_0 b_1) + \alpha b_0(-b_1 a_0 + a_1))$$

and $\alpha = \frac{\xi_0}{\xi_{-2}}$.

We prove by induction. When $n = 0$, we have the following:

If $i = 1$, then we get

$$\xi_{-1}\psi_1(0) = \xi_{-1} \frac{1 - b_0 b_1}{\mu - b_0 a_1 + a_0}$$

$$
\begin{aligned}
&= \xi_{-1}\frac{1-b_0b_1}{\frac{1}{\alpha}(-b_0(1-b_0b_1)+b_0(-b_1a_0+a_1)+(-b_0a_1+a_0)}\xi_{-1}\frac{1-b_0b_1}{\frac{1}{\alpha}(-b_0(1-b_0b_1)+a_0(1-b_0b_1))}\\
&= \xi_{-1}\frac{1-b_0b_1}{(1-b_0b_1)(-b_0\frac{1}{\alpha}+a_0)}\\
&= \xi_{-1}\frac{\alpha}{-b_0+a_0\alpha}\\
&= \frac{\xi_0\xi_{-1}}{a_0\xi_0-b_0\xi_{-2}}\\
&= \xi_1.
\end{aligned}
$$

Similarly, for $i=2$.

Now assume that $n \geq 1$. Then

$$
\begin{aligned}
\xi_{2n+1} &= \frac{\xi_{2n}\xi_{2n-1}}{a_{2n}\xi_{2n}-b_{2n}\xi_{2n-2}}\\
&= \frac{\xi_0\prod\limits_{j=0}^{n-1}\psi_2(j)\xi_{-1}\prod\limits_{j=0}^{n-1}\psi_1(j)}{a_0\xi_0\prod\limits_{j=0}^{n-1}\psi_2(j)-b_0\xi_0\prod_{j=0}^{n-2}\psi_2(j)}\\
&= \frac{a\xi_0\prod_{j=0}^{n-1}\psi_2(j)\xi_{-1}\prod\limits_{j=0}^{n-2}\psi_1(j)}{\xi_0\prod\limits_{j=0}^{n-2}\psi_2(j)(a_0\psi_2(n-1)-b_0)}\\
&= \frac{\psi_2(n-1)\xi_{-1}\prod\limits_{j=0}^{n-1}\psi_1(j)}{a_0\psi_2(n-1)-b_0}\\
&= \frac{\frac{1-b_0b_1}{(b_0b_1)^n\nu-b_1a_0+a_1}\xi_{-1}\prod\limits_{j=0}^{n-1}\psi_1(j)}{a_0\frac{1-b_0b_1}{(b_0b_1)^n\nu-b_1a_0+a_1}-b_0}\\
&= \frac{(1-b_0b_1)\xi_{-1}\prod\limits_{j=0}^{n-1}\psi_1(j)}{-b_0\nu(b_0b_1)^n-b_0a_1+a_0}.
\end{aligned}
$$

But as $\mu=-b_0\nu$, we get

$$
\begin{aligned}
\xi_{2n+1} &= \frac{(1-b_0b_1)\xi_{-1}\prod\limits_{j=0}^{n-1}\psi_1(j)}{\mu(b_0b_1)^n-b_0a_1+a_0} = \xi_{-1}\psi_1(n)\prod_{j=0}^{n-1}\psi_1(j)\\
&= \xi_{-1}\prod_{j=0}^{n}\psi_1(j).
\end{aligned}
$$

By similar argument, we can show that

$$
\xi_{2n+2} = \xi_0\prod_{j=0}^{n}\psi_2(j)
$$

and is omitted.

This completes the inductive proof. □

3. Behavior of the Well-Defined Solutions

Here, we discuss the behavior of Equation (2) using its closed-form well-defined solutions.

During subsections (3.1) and (3.2), we assume that

$$a_0 \neq b_0 a_1 \text{ and } a_1 \neq b_1 a_0.$$

3.1. Case $b_0 b_1 \neq 1$

Theorem 2. *Suppose that $\{\xi_n\}_{n=-2}^{\infty}$ is a well-defined solution of Equation (2). If $b_0 b_1 > 1$, then the solution $\{\xi_n\}_{n=-2}^{\infty}$ converges to zero.*

Proof. Assume that $b_0 b_1 > 1$. Then $\psi_t(j)$ converges to zero as $j \to \infty$, $t = 1, 2$. This implies that there exists $j_0 \in \mathbb{N}_0$, such that $| \psi_t(j) |< \epsilon$, for some $0 < \epsilon < 1$ for all $j \geq j_0$. Therefore, for $t = 1, 2$ we get

$$\begin{aligned} | \xi_{2m+t} | &= | \xi_{-2+t} || \prod_{j=0}^{m} \psi_t(j) | \\ &= | \xi_{-2+t} || \prod_{j=0}^{j_0-1} \psi_t(j) || \prod_{j=j_0}^{m} \psi_t(j) | \\ &< | \xi_{-2+t} || \prod_{j=0}^{j_0-1} \psi_t(j) | \epsilon^{m-j_0+1}. \end{aligned}$$

This implies that the solution $\{\xi_n\}_{n=-2}^{\infty}$ converges to zero. □

In the remainder of this subsection, we suppose that $b_0 b_1 < 1$. Then, we have the following cases:

1. $a_0 > b_0 a_1$ and $a_1 > b_1 a_0$;
2. $a_0 > b_0 a_1$ and $a_1 < b_1 a_0$;
3. $a_0 < b_0 a_1$ and $a_1 > b_1 a_0$.

The case $a_0 < b_0 a_1$ and $a_1 < b_1 a_0$ implies that $b_0 b_1 > 1$, which is a contradiction.

Let

$$A_1 = \frac{1 - b_0 b_1}{-b_0 a_1 + a_0} \text{ and } A_2 = \frac{1 - b_0 b_1}{-b_1 a_0 + a_1}.$$

Theorem 3. *Assume that $a_0 > b_0 a_1$, $a_1 > b_1 a_0$ and let $\{\xi_n\}_{n=-2}^{\infty}$ be a well-defined solution of Equation (2). We have the following:*

1. *If $a_0 > 1 - b_0 b_1 + b_0 a_1$ and $a_1 > 1 - b_0 b_1 + b_1 a_0$, then $\{\xi_n\}_{n=-2}^{\infty}$ converges to 0 as $n \to \infty$.*
2. *If $b_0 a_1 < a_0 < 1 - b_0 b_1 + b_0 a_1$ and $b_1 a_0 < a_1 < 1 - b_0 b_1 + b_1 a_0$, then $\xi_{2n+1} \to \infty(sgn(\xi_{-1}))$ and $\xi_{2n} \to \infty(sgn(\xi_0))$ as $n \to \infty$.*
3. *If $a_0 > 1 - b_0 b_1 + b_0 a_1$ and $b_1 a_0 < a_1 < 1 - b_0 b_1 + b_1 a_0$, then $\xi_{2n+1} \to 0$ and $\xi_{2n} \to \infty(sgn(\xi_0))$ and as $n \to \infty$.*
4. *If $b_0 a_1 < a_0 < 1 - b_0 b_1 + b_0 a_1$ and $a_1 > 1 - b_0 b_1 + b_1 a_0$, then $\xi_{2n+1} \to \infty(sgn(\xi_{-1}))$ and $\xi_{2n} \to 0$ as $n \to \infty$.*

Proof. When $b_0 b_1 < 1$, we get $\psi_i(j) \to A_i$, $i = 1, 2$.

As $b_0b_1 < 1$, we have
$$|A_1| = \frac{1-b_0b_1}{-b_0a_1+a_0} \text{ and } |A_2| = \frac{1-b_0b_1}{-b_1a_0+a_1}.$$

1. The given conditions imply that $|A_t| < 1$ for $t = 1, 2$. So that there exists $j_0 \in \mathbb{N}_0$, such that $\psi_t(j) < \epsilon$ for a given $0 < \epsilon < 1$, $t = 1, 2$.

   Then for $t = 1, 2$ we get
$$\begin{aligned} | \xi_{2m+t} | &=| \xi_{-2+t} || \prod_{j=0}^{m} \psi_t(j) | \\ &=| \xi_{-2+t} || \prod_{j=0}^{j_0-1} \psi_t(j) || \prod_{j=j_0}^{m} \psi_t(j) | \\ &<| \xi_{-2+t} || \prod_{j=0}^{j_0-1} \psi_t(j) | \epsilon^{m-j_0+1}. \end{aligned}$$

   Therefore, the solution $\{\xi_n\}_{n=-2}^{\infty}$ converges to zero.

2. The given conditions imply that $|A_t| > 1$ for $t = 1, 2$. By the same argument as in (1), we get the result.

Using (1) and (2), we get (3) and (4). □

Theorem 4. *Assume that $a_0 > b_0a_1$, $a_1 < b_1a_0$ and let $\{\xi_n\}_{n=-2}^{\infty}$ be a well-defined solution of Equation (2). We have the following:*

1. *If $a_0 > \max\{1 - b_0b_1 + b_0a_1, \frac{1}{b_1}(1 - b_0b_1 + a_1)\}$, then $\{\xi_n\}_{n=-2}^{\infty}$ converges to 0 as $n \to \infty$.*
2. *If $a_0 < \min\{1 - b_0b_1 + b_0a_1, \frac{1}{b_1}(1 - b_0b_1 + a_1)\}$, then $\xi_{2n+1} \to \infty(sgn(\xi_{-1}))$ and $\xi_{2n} \to \infty(sgn(\xi_0))$ as $n \to \infty$.*
3. *If $1 - b_0b_1 + b_0a_1 < a_0 < \frac{1}{b_1}(1 - b_0b_1 + a_1)$, then $\xi_{2n+1} \to 0$ and $\xi_{2n} \to \infty(sgn(\xi_0))$ as $n \to \infty$.*
4. *If $\frac{1}{b_1}(1 - b_0b_1 + a_1) < a_0 < 1 - b_0b_1 + b_0a_1$, then $\xi_{2n+1} \to \infty(sgn(\xi_{-1}))$ and $\xi_{2n} \to 0$ as $n \to \infty$.*

Proof. When $b_0b_1 < 1$, we have $\psi_i(j) \to A_i$, $i = 1, 2$.

It is enough to see that the conditions $a_0 > b_0a_1$ and $a_1 < b_1a_0$ imply that
$$|A_1| = \frac{1-b_0b_1}{-b_0a_1+a_0} \text{ and } |A_2| = \frac{1-b_0b_1}{b_1a_0-a_1}.$$

□

Theorem 5. *Assume that $a_0 < b_0a_1$, $a_1 > b_1a_0$ and let $\{\xi_n\}_{n=-2}^{\infty}$ be a well-defined solution of Equation (2). We have the following:*

1. *If $a_1 > \max\{1 - b_0b_1 + b_1a_0, \frac{1}{b_0}(1 - b_0b_1 + a_0)\}$, then $\{\xi_n\}_{n=-2}^{\infty}$ converges to 0 as $n \to \infty$.*
2. *If $a_1 < \min\{1 - b_0b_1 + b_1a_0, \frac{1}{b_0}(1 - b_0b_1 + a_0)\}$, then $\xi_{2n+1} \to \infty(sgn(\xi_{-1}))$ and $\xi_{2n} \to \infty(sgn(\xi_0))$ as $n \to \infty$.*
3. *If $1 - b_0b_1 + b_1a_0 < a_1 < \frac{1}{b_1}(1 - b_0b_1 + a_0)$, then $\xi_{2n+1} \to \infty(sgn(\xi_{-1}))$ and $\xi_{2n} \to 0$ as $n \to \infty$.*
4. *If $\frac{1}{b_1}(1 - b_0b_1 + a_0) < a_1 < 1 - b_0b_1 + b_1a_0$, then $\xi_{2n+1} \to 0$ and $\xi_{2n} \to \infty(sgn(\xi_0))$ as $n \to \infty$.*

Proof. It is enough to see that the conditions $a_0 < b_0a_1$ and $a_1 > b_1a_0$ imply that
$$|A_1| = \frac{1-b_0b_1}{b_0a_1-a_0} \text{ and } |A_2| = \frac{1-b_0b_1}{-b_1a_0+a_1}.$$

□

In the following two results, we suppose that the conditions

$$a_0 = 1 - b_0 b_1 + b_0 a_1 \text{ and } a_1 = 1 - b_0 b_1 + b_1 a_0 \tag{5}$$

are satisfied.

Theorem 6. *All well-defined solutions of Equation (2) satisfy Conditions (5) converge to a period-2 solution.*

Proof. The conditions

$$a_0 = 1 - b_0 b_1 + b_0 a_1 \text{ and } a_1 = 1 - b_0 b_1 + b_1 a_0$$

imply that $\psi_t(j)$ converges to 1 as $j \to \infty$, $t = 1, 2$. This implies that there exists $j_0 \in \mathbb{N}_0$, such that $\psi_t(j) > 0$, for all $j \geq j_0$ and $t = 1, 2$. Therefore, for $t = 1, 2$, we get

$$\begin{aligned} \xi_{2m+t} &= \xi_{-2+t} \prod_{j=0}^{m} \psi_t(j) \\ &= \xi_{-2+t} \prod_{j=0}^{j_0-1} \psi_t(j) \prod_{j=j_0}^{m} \psi_t(j) \\ &= \xi_{-2+t} \prod_{j=0}^{j_0-1} \psi_t(j) \sum_{j=j_0}^{m} \ln(\psi_t(j)). \end{aligned}$$

We shall test the convergence of the series $\sum_{j=j_0}^{\infty} \ln \psi_t(j)$.

Using convergence criteria rules we find that the series $\sum_{j=j_0}^{\infty} \ln \psi_t(j)$ is convergent.

Suppose that $\sum_{j=j_0}^{\infty} \ln \psi_t(j) = \theta_t$, $t = 1, 2$.

This implies that there exist two real numbers ρ_t such that

$$\lim_{m \to \infty} \xi_{2m+t} = \rho_t, \quad t \in \{1, 2\}.$$

Therefore, $\{\xi_n\}_{n=-2}^{\infty}$ converges to the period-2 solution

$$\{..., \rho_1, \rho_2, \rho_1, \rho_2, ...\},$$

where

$$\rho_t = \xi_{-2+t}(\prod_{j=0}^{j_0-1} \psi_t(j))\theta_t, \quad t = 1, 2.$$

This completes the proof. □

Theorem 7. *Let $\{\xi_n\}_{n=-2}^{\infty}$ be a well-defined solution of Equation (2) that satisfies Conditions (5). If $\alpha = \dfrac{1 - b_0 b_1}{-b_1 a_0 + a_1}$, then $\{\xi_n\}_{n=-2}^{\infty}$ is a period-2 solution.*

Proof. If $\alpha = \dfrac{1-b_0b_1}{-b_1a_0+a_1}$, then $\nu = 0$ and $\mu = 0$. Then

$$\psi_1(j) = \frac{1-b_0b_1}{-b_0a_1+a_0} = A_1 \text{ and } \psi_2(j) = \frac{1-b_0b_1}{-b_1a_0+a_1} = A_2.$$

The conditions

$$a_0 = 1 - b_0b_1 + b_0a_1 \text{ and } a_1 = 1 - b_0b_1 + b_1a_0$$

imply that

$$\psi_1(j) = 1 = \psi_2(j).$$

Then

$$\xi_{2m+t} = \xi_{-2+t}\prod_{j=0}^{m}\psi_t(j) = \xi_{-2+t}, \; t = 1,2 \text{ and } m = 0,1,....$$

The proof is completed. □

3.2. Case $b_0b_1 = 1$

This subsection is devoted to studing a special case of Equation (2) when $b_0b_1 = 1$.

In this case, using simple calculations, we get the equations

$$r_{2n+2} = r_{2n} - b_1a_0 + a_1$$

and

$$r_{2n+3} = r_{2n+1} - b_0a_1 + a_0.$$

This implies that

$$\xi_n = \begin{cases} \xi_{-1}\prod_{j=0}^{\frac{n-1}{2}}\gamma_1(j) & ,n = 1,3,5,..., \\ \xi_0\prod_{j=0}^{\frac{n-2}{2}}\gamma_2(j) & ,n = 2,4,6,..., \end{cases} \tag{6}$$

where

$$\gamma_1(j) = \frac{\alpha}{-b_0 + a_0\alpha + \alpha(-b_0a_1 + a_0)j}$$

and

$$\gamma_2(j) = \frac{\alpha}{1 + \alpha(-b_1a_0 + a_1)(j+1)}$$

where $\alpha = \frac{\xi_0}{\xi_{-2}}$.

Theorem 8. *Every well-defined solution of Equation (2) converges to zero.*

Proof. The proof is similar to that of Theorem (2) and is omitted. □

3.3. Case $a_0 = b_0a_1$ and $a_1 = b_1a_0$

The conditions $a_0 = b_0a_1$ and $a_1 = b_1a_0$ imply that $b_0b_1 = 1$.

In this case, we have

$$r_{2n+2} = r_{2n} \text{ and } r_{2n+3} = r_{2n+1}.$$

This implies that

$$\xi_n = \begin{cases} \xi_{-1}\left(\dfrac{\alpha}{a_0\alpha - b_0}\right)^{\frac{n+1}{2}} & ,n = 1,3,5,..., \\ \xi_0\alpha^{\frac{n}{2}} & ,n = 2,4,6,..., \end{cases} \tag{7}$$

where $\alpha = \frac{\xi_0}{\xi_{-2}}$.

Theorem 9. *Assume that* $a_0 = b_0 a_1$, $a_1 = b_1 a_0$ *and let* $\{\xi_n\}_{n=-2}^{\infty}$ *be a well-defined solution of Equation* (2) *such that* $a_0 > b_0 + 1$. *If* $\frac{b_0}{a_0-1} < \alpha < 1$, *then* $\{\xi_n\}_{n=-2}^{\infty}$ *converges to zero.*

Proof. Condition $\frac{b_0}{a_0-1} < \alpha < 1$ implies that $\alpha > \frac{b_0}{a_0}$, from which $a_0\alpha - b_0 > 0$ and $\frac{\alpha}{a_0\alpha - b_0} < 1$. Then $\xi_{2n} \to 0$ and $\xi_{2n+1} \to 0$ as $n \to \infty$ and this completes the proof. □

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ORCID

Raafat Abo-Zeid https://orcid.org/0000-0002-1858-5583
Mehmet Gümüş https://orcid.org/0000-0002-7447-479X
Ana Catarina Carapito https://orcid.org/0000-0002-5868-6266
Abdul Qadeer Khan https://orcid.org/0000-0002-0278-1352

References

[1] S. Elaydi, *An Introduction to Difference Equations*, 2nd ed., Springer-Verlag, New York, 1999.

[2] V.L. Kocic and G. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications*, Kluwer Academic, Dordrecht, 1993. [Web]

[3] M. R. S. Kulenović and G. Ladas, *Dynamics of Second Order Rational Difference Equations: With Open Problems and Conjectures*, Chapman and Hall/CRC, Boca Raton, 2002.

[4] R.E. Mickens, *Difference Equations: Theory, Applications and Advanced Topics*, 3rd ed., CRC Press, Boca Raton, 2015.

[5] R. Abo-Zeid, *Behavior of solutions of a second order rational difference equation*, Math. Morav., **23**(1) (2019), 11–25. [CrossRef]

[6] Ö. Aktaş, M. Kara and Y. Yazlık, *On a solvable difference equations system*, Filomat, **39**(7) (2025), 2229–2267. [CrossRef] [Scopus] [Web of Science]

[7] Ö. Aktaş, M. Kara and Y. Yazlık, *Solvability of a two-dimensional system of difference equations with constant coefficients*, Ikonion J. of Math., **6**(2) (2024), 1–12. [CrossRef]

[8] M. Aloqeili, *Dynamics of a rational difference equation*, Appl. Math. Comput., **176**(2) (2006), 768–774. [CrossRef] [Scopus] [Web of Science]

[9] A.M. Amleh, E. Camouzis and G. Ladas, *On the dynamics of a rational difference equation, Part 2*, Int. J. Difference Equ., **3**(2) (2008), 195–225. [Web]

[10] A.M. Amleh, E. Camouzis and G. Ladas, *On the dynamics of a rational difference equation, Part 1*, Int. J. Difference Equ., **3**(1) (2008), 1–35. [Web]

[11] A. Anisimova and I. Bula, *Some problems of second-order rational difference equations with quadratic terms*, Int. J. Difference Equ., **9**(1) (2014), 11–21. [Web]

[12] M. Dehghan, C.M. Kent, R. Mazrooei-Sebdani, N.L. Ortiz and H. Sedaghat, *Dynamics of rational difference equations containing quadratic terms*, J. Difference Equ. Appl., **14**(2) (2008), 191–208. [CrossRef] [Scopus] [Web of Science]

[13] S. Devecioğlu and M. Kara, *Solvability of a four-dimensional system of difference equations*, Eskişehir Tech. Univ. J. Sci. Tech. B-Theor. Sci., **12**(1) (2024), 62–74. [CrossRef]

[14] M. Gümüş and R. Abo-Zeid, *Global behavior of a rational second order difference equation*, J. Appl. Math., **62**(1-2) (2020), 119–133[CrossRef] [Scopus] [Web of Science]

[15] M. Kara and Y. Yazlık, *On a class of difference equations system of fifth-order*, Fund. J. Math. Appl., **7**(3) (2024), 186–202. [CrossRef]

[16] M.R.S. Kulenović and M. Mehuljić, *Global behavior of some rational second order difference equations*, Int. J. Difference Equ., **7**(2) (2012), 153–162. [Web]

[17] S. Stevic, *More on a rational recurrence relation*, Appl. Math. E-Notes, **4** (2004), 80–84. [Scopus]

[18] C. Çınar, *On the positive solution of the difference equation* $x_{n+1} = \frac{ax_{n-1}}{1+bx_n x_{n-1}}$, Appl. Math. Comput., **156** (2004), 587–590. [CrossRef] [Scopus]

[19] R. Abo-Zeid, *Global behavior of a rational difference equation with quadratic term*, Math. Morav., **18**(1) (2014), 81–88. [Web]

[20] R. Abo-Zeid, *Global behavior of two third order rational difference equations with quadratic terms*, Math. Slovaca, **69**(1) (2019), 147–158. [CrossRef] [Scopus] [Web of Science]

[21] R. Abo-Zeid, *Global behavior of a fourth order difference equation with quadratic term*, Bol. Soc. Mat. Mexicana, **25**(1) (2019), 187–194. [CrossRef] [Scopus] [Web of Science]

[22] R. Abo-Zeid, *Behavior of solutions of a higher order di erence equation*, Alabama J. Math., **42** (2018), 1–10. [Web]

[23] R. Abo-Zeid, *On the solutions of a higher order difference equation*, Georgian Math. J., **27**(2) (2020), 165-175. [CrossRef]

[24] R. Abo-Zeid, *Forbidden set and well-defined solutions of a higher order difference equation*, Dyn. Contin. Discrete Impuls. Syst. Ser. B Appl. Algorithms, **25** (2018), 75–84. [Web]

[25] R. Abo-Zeid, *Forbidden sets and stability in some rational difference equations*, J. Difference Equ. Appl., **24**(2) (2018), 220–239. [CrossRef] [Scopus] [Web of Science]

[26] R. Abo-Zeid, *Global behavior of a higher order rational difference equation*, Filomat, **30**(12) (2016), 3265–3276. [CrossRef] [Scopus] [Web of Science]

[27] R. Abo-Zeid, *Global behavior of a third order rational difference equation*, Math. Bohem., **139**(1) (2014), 25–37. [Scopus] [Web of Science]

[28] R. Abo-Zeid, *Global behavior of a fourth order difference equation*, Acta Commentationes Univ. Tartuensis Math., **18**(2) (2014), 211–220. [CrossRef] [Scopus] [Web of Science]

[29] R. Abu-Saris, C. Çınar and İ. Yalçınkaya, *On the asymptotic stability of* $x_{n+1} = (a + x_n x_{n-k})/(x_n + x_{n-k})$, Comput. Math. Appl., **56**(5) (2008), 1172–1175. [CrossRef] [Scopus] [Web of Science]

[30] F. Balibrea and A. Cascales, *On forbidden sets*, J. Difference Equ. Appl., **21**(10) (2015), 974–996. [CrossRef] [Scopus] [Web of Science]

[31] M. Gümüş and R. Abo-Zeid, *On the well-defined solutions of a* $(2k+2)$*th order difference equation*, Dyn. Contin. Discrete Impuls. Syst. Ser. B Appl. Algorithms, **25** (2018), 129–143. [Web]

[32] C.M. Kent and H. Sedaghat, *Global attractivity in a quadratic-linear rational difference equation with delay*, J. Difference Equ. Appl., **15**(10) (2009), 913–925. [CrossRef] [Scopus] [Web of Science]

[33] H. Sedaghat, *On third order rational equations with quadratic terms*, J. Difference Equ. Appl., **14**(8) (2008), 889–897. [CrossRef]

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