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*Research Paper*

# A Note on Fractional Parametric-Type Laplace Transforms

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## Abstract

In recent articles an extension of the exponential function including one or several parameters have been exploited to introduce generalized forms of linear dynamical systems, including population dynamics models, and some graphical curves and Chebyshev functions. In this article, by means of the Blissard problem we define the reciprocal of parametric or fractional parametric-type exponentials in order to define new-type Laplace transforms. Some examples are shown, derived by the second author using the computer algebra system Mathematica©.

**Key Words:** Parametric exponential functions; Laguerre-type exponentials; Generalized Blissard problem; Laplace transform theory

**AMS 2020 Classification:** 11B73; 26A33; 44A10

## 1. Introduction

It is well known that the exponential function behaves as an eigenfunction of the differentiation operator, since

$$De^{cx} = c\,e^{cx}$$

(where $x$ is a real or complex variable, $D := d/dx$, and $c$ is real or complex number). Likewise the *Laguerre-type exponential*

$$e_1(x) := \sum_{k=0}^{\infty} \frac{x^k}{(k!)^2} \tag{1}$$

is an eigenfunction of the so called *Laguerre derivative*,

$$\hat{D}_L := DxD = D + x\,D^2\,,$$

since

$$\hat{D}_L\; e_1(cx) = c\,e_1(cx)\,.$$

The above result has been extended as follows [1]-[8].

Consider the differential operator, containing $n+1$ derivatives

$$\begin{aligned} \hat{D}_{(n-1)L} &:= Dx\cdots DxDxD \\ &= D\left(xD + x^2D^2 + \cdots + x^{n-1}D^{n-1}\right) \\ &= S(n,1)D + S(n,2)xD^2 + \cdots + S(n,n)x^{n-1}D^n, \end{aligned}$$

where $S(n,k)$, $(k=1,2,\ldots,n)$, are the Stirling numbers of the second kind, and the function

$$e_n(x) := \sum_{k=0}^{\infty} \frac{x^k}{(k!)^{n+1}}.$$

We have proven in [1] that the function $e_n(c\,x)$ is an eigenfunction of the operator $\hat{D}_{nL}$, that is

$$\hat{D}_{nL}\; e_n(c\,x) = c\,e_n(c\,x).$$

**Remark 1.** For completeness, we recall that the operators $\hat{D}_L = DxD$ and its iterates as $\hat{D}_{nL} = DxDxDx\cdots DxD$ can be considered as particular cases of the hyper-Bessel differential operators when $\alpha_0 = \alpha_1 = \cdots = \alpha_n = 1$ (the special case considered in operational calculus by Ditkin and Prudnikov [9]). In general, the *Bessel-type differential operators of arbitrary order* $n$ were introduced by Dimovski, in 1966 [10] and later called by Kiryakova *hyper-Bessel operators*, because are closely related to their eigenfunctions, called hyper-Bessel by Delerue [11], in 1953. These operators were studied in 1994 by Virginia Kiryakova in her book [2], Ch. 3.

## 2. The Parametric Case

In a preceding article [1], we have proven the result

**Theorem 1.** *The function of the complex variable* $x$

$$e_{1,m}(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!\,(k+m)!}\,, \tag{2}$$

(*where* $m$ *is a positive integer number*) *is an eigenfunction of the operator*

$$DxD + mD\,, \tag{3}$$

*since for every real or complex constant* $c$*, we find*

$$(DxD + mD)\,e_{1,m}(c\,x) = c\,e_{1,m}(c\,x)\,.$$

*Proof.* In fact

$$(DxD + mD)\sum_{k=0}^{\infty}\frac{c^k\,x^k}{k!\,(k+m)!} \quad = \quad c\left[D\,x\sum_{k=1}^{\infty}\frac{c^{k-1}\,x^{k-1}}{(k-1)!\,(k+m)!} + m\sum_{k=1}^{\infty}\frac{c^{k-1}\,x^{k-1}}{(k-1)!\,(k+m)!}\right]$$

$$
\begin{aligned}
&= c\left[D\sum_{k=0}^{\infty}\frac{c^k\,x^{k+1}}{k!\,(k+m+1)!}+m\sum_{k=0}^{\infty}\frac{c^k\,x^k}{k!\,(k+m+1)!}\right] \\
&= c\left[\sum_{k=0}^{\infty}(k+1)\frac{c^k\,x^k}{k!\,(k+m+1)!}+m\sum_{k=0}^{\infty}\frac{c^k\,x^k}{k!\,(k+m+1)!}\right] \\
&= c\sum_{k=0}^{\infty}(k+m+1)\frac{c^k\,x^k}{k!\,(k+m+1)!}=c\sum_{k=0}^{\infty}\frac{c^k\,x^k}{k!\,(k+m)!}\,.
\end{aligned}
$$

□

## 2.1. The fractional case

For any real number $\alpha > 0$, the fractional derivative of powers, according the Euler definition, falling as a special case in the definition of fractional derivative introduced by Caputo, writes

$$
D_x^\alpha x^n = \begin{cases} \dfrac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}\, x^{n-\alpha}, & \text{if} \quad n > \lceil\alpha\rceil - 1\,, \\[2ex] 0\,, \quad \text{if} \quad n = 0, 1, \ldots, \lceil\alpha\rceil - 1\,, \end{cases}
$$

where $n \geq 0$ and $\lceil\alpha\rceil$ denotes the ceiling function, that is the smallest integer greater than or equal to $\alpha$. If $c$ is a constant then $D_x^\alpha c = 0$.

**Remark 2.** We recall that the Caputo derivative is defined as

$$
D_{a+}^\alpha f(x) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_a^x \frac{f^{(m)}(\tau)}{(x-\tau)^{\alpha-m+1}}\, d\tau\,, \quad \text{where } m = \lceil\alpha\rceil\,, \text{ if } \alpha \notin \mathbb{N} \\[2ex] f^{(\alpha)}(x), \quad \text{if} \quad \alpha \in \mathbb{N}\,, \end{cases}
$$

and reduces to the preceding equation when $a = 0$ and $f(x) = x^n$.

The fractional exponential function (depending on $\alpha$) is defined as

$$
\mathrm{Exp}_\alpha(t) = 1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots + \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} + \ldots\,.
$$

It is an eigenfunction of the operator $D_x^\alpha$, since it results

$$
D_x^\alpha \mathrm{Exp}_\alpha(xt) = t^\alpha\, \mathrm{Exp}_\alpha(xt)\,.
$$

More generally, we can consider the fractional parametric-type exponential function defined as

$$
\mathrm{Exp}_{\alpha,m}(x) = \sum_{k=0}^{\infty} \frac{x^{k\alpha}}{\Gamma[k\alpha+1]\,\Gamma[k\alpha+m+1]}\,, \tag{4}
$$

and the operator

$$
D_x^\alpha\, x^\alpha D_x^\alpha + m\, D_x^\alpha\,, \tag{5}
$$

for which it results

$$
(D_x^\alpha\, x^\alpha D_x^\alpha + m\, D_x^\alpha)\ [\mathrm{Exp}_{\alpha,m}(tx)] = t^\alpha\, [\mathrm{Exp}_{\alpha,m}(tx)]\,.
$$

The purpose of this article is to introduce some generalized types of the Laplace transform including an integer parameter $m \geq 0$, and the operator $\hat{D}_{x,m} = DxD + mD$ in Theorem 2.1, or its fractional version in equation (5).

We use an expansion of the type

$$g(x,m) = \sum_{k=0}^{\infty} a_k(m) x^k \,,$$

with real or complex $a_k(m)$ coefficients, converging in all plane and satisfying the eigenvalue property

$$\hat{D}_{x,m}\, g(c\,x,m) = \lambda(c)\, g(x,m)\,,$$

with respect to the operator (3) or (5).

In what follows we use the compact notation for the coefficients of the $g(x,m)$ expansion, letting $a_m := \{a_{k,m}\}\,, (k = 0,1,2,\dots)$, a sequence which identifies the function $g(x,m)$.

After constructing the reciprocal $[g(x,m)]^{-1}$ of the considered expansion, we substitute this reciprocal function in the place of $\exp(-xs)$ in the Laplace transformation.

So we get the new Laplace-type transform

$$\mathcal{L}_{g(x,m)}(f) := \int_0^{+\infty} [g(xs,m)]^{-1} f(x)\, dx = \mathcal{F}_{g(x,m)}(s)\,.$$

In the one parameter case, we construct the reciprocal of the functions in (2) and (4).

An analogous result could be obtained in the multi-parameter case, but the relevant equations are more involved.

We exploit the reciprocal of these unusual exponentials, obtained using an extension of the Blissard problem, to construct generalized forms of the Laplace transform.

In all cases a generalized type of Laplace transform can be defined, and some numerical check is performed using the computer algebra program Mathematica©.

## 3. The Reciprocal of a Power Series

It is well known that, using the Blissard problem [13], the coefficients of the reciprocal of a power series are expressed in terms of Bell polynomials.

### 3.1. The reciprocal of the parametric-type exponential (2).

Given the sequence $a := \{a_k\} = (a_0, a_1, a_2, a_3, \dots)$, we consider the function

$$\frac{1}{a_0 + a_1 x + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \dots} \qquad (x \geq 0),$$

in which we assume $a_m := \{a_{k,m}\} = \{1/(k+m)!\}$, that is

$$\frac{1}{\frac{1}{m!} + \frac{x}{(1+m)!} + \frac{x^2}{2!(2+m)!} + \frac{x^3}{3!(3+m)!} + \dots} \qquad (x \geq 0). \tag{6}$$

When $m = 0$, the reciprocal of the Laguerre-type exponential function (1) is recovered.

Note that the functions (6) are complete monotonic functions decreasing from the initial value 1, at $x = 0$, and vanishing at infinity.

Therefore, using the umbral formalism (that is, letting $a_{k,m} \equiv a^{k,m}$ and $b_{k,m} \equiv b^{k,m}$), and exploting the Blissard problem [13], from the equation

$$e_{1,m}[a_m(s\,x)]\,e_{1,m}[b_m(s\,x)] = 1\,,$$

in which the coefficieents are depending on a given real or complex variable $s$, we find

$$\frac{1}{\displaystyle\sum_{k=0}^{\infty} \frac{a_{k,m}(s\,x)^k}{k!}} = \sum_{k=0}^{\infty} \frac{b_{k,m}(s\,x)^k}{k!}\,,$$

so that we introduce the definition

**Definition 1.** The parametric Laguerre-type Laplace transform, where the $a_{k,m}$ are chosen according to equation *(6)*, is defined as

$$\mathcal{L}_{g(x,m)}(f) := \int_0^{\infty} \frac{f(x)}{\displaystyle\sum_{k=0}^{\infty} \frac{a_{k,m}(s\,x)^k}{k!}}\,dx = \int_0^{\infty} f(x) \sum_{k=0}^{\infty} \frac{b_{k,m}(s\,x)^k}{k!}\,dx = \mathcal{F}_{g(x,m)}(s)\,,$$

where the $b_{k,m}$ coefficients are given by

$$\begin{cases} b_{0,m} := m!, \\ \\ b_{n,m} = Y_n\left(-1!, \frac{1}{(1+m)!}; 2!, \frac{1}{(2+m)!}; -3!, \frac{1}{(3+m)!}; \ldots; (-1)^n n!, \frac{1}{(n+m)!}\right), \quad (\forall\, n > 0), \end{cases} \tag{7}$$

and $Y_n$ is the $n$th Bell polynomial *[13]*.

Using the Faà di Bruno formula, the second equation in (7), $\forall\, n \geq 0$, writes

$$b_{n,m} = \frac{(n+m)!}{n!} \sum_{k=0}^{n} (-1)^k\, k!\, a_0^{-(k+1)}\, B_{n,k}\left(\frac{1!}{(1+m)!}, \frac{2!}{(2+m)!}, \ldots, \frac{(n-k+1)!}{(n-k+1+m)!}\right),$$

where $B_{n,k}$ are partial Bell polynomials [12, 13].

### 3.2. The reciprocal of the fractional parametric-type exponential (4).

We consider the function

$$\frac{1}{a_{0,m} + a_{1,m}\dfrac{x^{\alpha}}{\Gamma(\alpha+1)} + a_{2,m}\dfrac{x^{2\alpha}}{\Gamma(2\alpha+1)} + a_{3,m}\dfrac{x^{3\alpha}}{\Gamma(3\alpha+1)} + \ldots} \qquad (x \geq 0),$$

assuming $a_m := \{a_{k,m}\} = \{1/\Gamma(k\alpha + m + 1)\}$, that is

$$\frac{1}{\frac{1}{\Gamma(m+1)} + \dfrac{x^{\alpha}}{\Gamma(\alpha+1)\,\Gamma(\alpha+m+1)} + \dfrac{x^{2\alpha}}{\Gamma(2\alpha+1)\,\Gamma(2\alpha+m+1)} + \dfrac{x^{3\alpha}}{\Gamma(3\alpha+1)\,\Gamma(3\alpha+m+1)} + \ldots} \qquad (x \geq 0),$$

the solution of the umbral equation

$$\mathrm{Exp}_{\alpha,m}[a_m(s\,x)]\,\mathrm{Exp}_{\alpha,m}[b_m(s\,x)] = 1\,,$$

in terms of the unknown sequence $\{b_{n,m}\}$, using the Blissard problem and Bell's polynomials, $\forall\, n > 0$, writes

$$\begin{cases} b_{0,m} := \Gamma(m+1), \\ b_{n,m} = Y_n(-1!, \frac{1}{\Gamma(\alpha+m+1)}; 2!, \frac{1}{\Gamma(2\alpha+m+1)}; -3!, \frac{1}{\Gamma(3\alpha+m+1)}; \ldots; (-1)^n n!, \frac{1}{\Gamma(n\alpha+m+1)}), \end{cases}$$

where $Y_n$ is the $n$th Bell polynomial [13].

In the particular case of the reciprocal of the $\mathrm{Exp}_{\alpha,m}(x)$ function we must assume $\{a\} \equiv \frac{1}{\Gamma(m+1)}\{1, 1, 1, \ldots\}$ and therefore, recalling that $B_{n,h}(1, 1, \ldots, 1) \equiv S_{n,h}$, that is the Stirling numbers of the second kind, it results

$$\begin{aligned} [\mathrm{Exp}_{\alpha,m}(x)]^{-1} &= 1 + \sum_{n=1}^{\infty}\sum_{h=1}^{n}(-1)^h \frac{\Gamma(h\alpha+m+1)}{\Gamma(m+1)} B_{n,h}(1, 1, \ldots, 1)\, x^{n\alpha} \\ &= \sum_{n=0}^{\infty}\sum_{h=0}^{n}(-1)^h \frac{\Gamma(h\alpha+m+1)}{\Gamma(m+1)} S_{n,h} \frac{x^{n\alpha}}{\Gamma(n\alpha+m+1)}, \end{aligned}$$

where we have put $S_{0,0} := 1$.

### 3.3. A fractional-type Laplace transform

Using the above definition for the reciprocal of the fractional exponential, we can introduce a fractional version (of order $\alpha$, $\alpha > 0$) of the Laplace Transform, by setting

$$\mathcal{L}_{\alpha,m}(f) := \int_0^{\infty} f(x)\, [\mathrm{Exp}_{\alpha,m}(sx)]^{-1}\, dx = \mathcal{F}_{\alpha,m}(s) =$$

$$= \int_0^{\infty} f(x) \left( \sum_{n=0}^{\infty}\sum_{h=0}^{n}(-1)^h \frac{\Gamma(h\alpha+m+1)}{\Gamma(m+1)} S_{n,h} \frac{x^{n\alpha}}{\Gamma(n\alpha+m+1)} \right) dx.$$

In what follows, we make a comparison among the classical Laplace Transform of assigned functions and the fractional order Laplace transforms of order $\alpha = 1/2$ and $\alpha = 3/2$.

As it is shown in the obtained results, in all cases the graphs of the modulus and argument of the ordinary Laplace Transform lies between the corresponding graphs of the two considered fractional order Laplace transforms. This provides a graphical evidence of the monotonicity property satisfied by the fractional order Laplace transforms.

## 4. Numerical Examples

### 4.1. Example 1. (m=0)

Assuming $m = 0$, consider the fractional Laplace Transforms $\mathcal{F}_{1/2} = \mathcal{F}_{0,1/2}$, $\mathcal{F}_{3/2} = \mathcal{F}_{0,3/2}$ of the Bessel function $J_0(2\sqrt{t})$, compared with the classical LT $\mathcal{F} = \mathcal{F}_{0,1}$ of the same function.

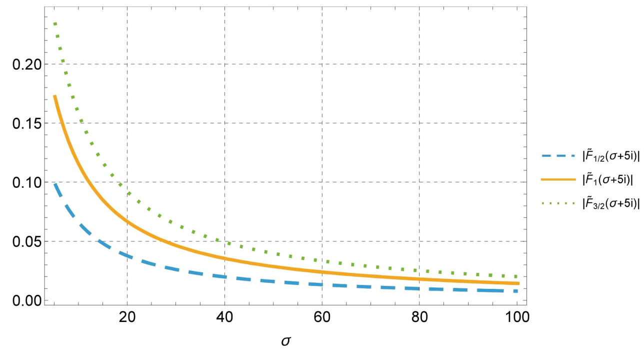


**Fig. 1.** Comparing the fractional LTs $\mathcal{F}_{0,1/2}$, $\mathcal{F}_{0,1}, \mathcal{F}_{0,3/2}$, of the function $J_0(2\sqrt{t})$ - the case of the modulus, assuming $s = \sigma + 5i$

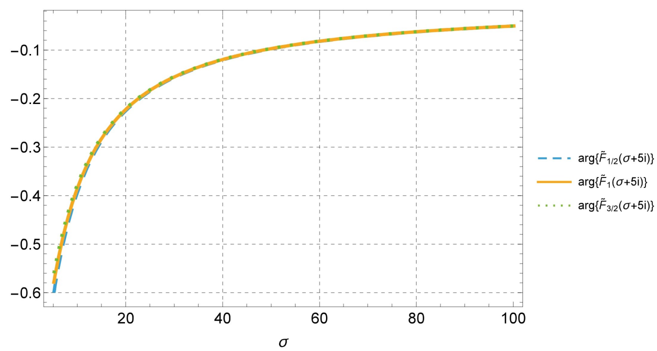


**Fig. 2.** Comparing the fractional LTs $\mathcal{F}_{0,1/2}$, $\mathcal{F}_{0,1}, \mathcal{F}_{0,3/2}$, of the function $J_0(2\sqrt{t})$ - the case of the argument, assuming $s = \sigma + 5i$

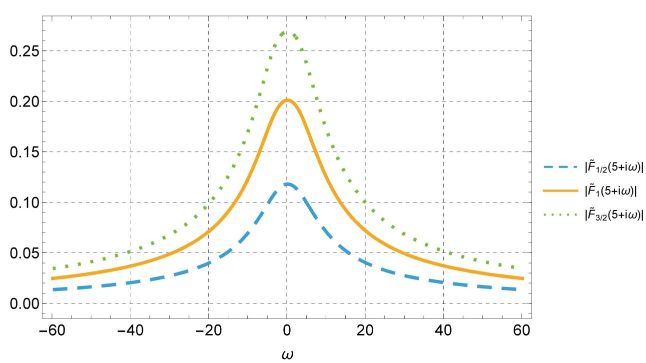


**Fig. 3.** Comparing the fractional LTs $\mathcal{F}_{0,1/2}$, $\mathcal{F}_{0,1}, \mathcal{F}_{0,3/2}$, of the function $J_0(2\sqrt{t})$ - the case of the modulus, assuming $s = 5 + i\,\omega$

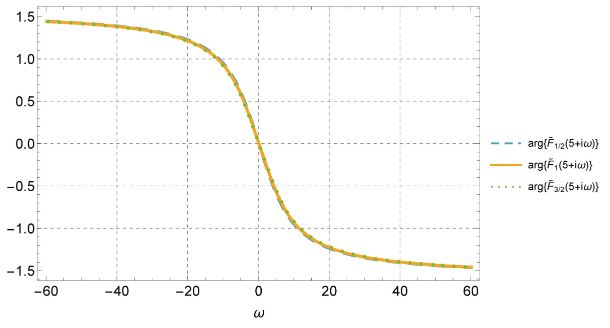


**Fig. 4.** Comparing the fractional LTs $\mathcal{F}_{0,1/2}$, $\mathcal{F}_{0,1}, \mathcal{F}_{0,3/2}$, of the function $J_0(2\sqrt{t})$ - the case of the argument, assuming $s = 5 + i\,\omega$

## 4.2. Example 1. (m=2)

Assuming $m = 2$, consider the fractional Laplace Transforms $\mathcal{F}_{1/2} = \mathcal{F}_{2,1/2}$, $\mathcal{F}_{3/2} = \mathcal{F}_{2,3/2}$ of the Bessel function $J_0(2\sqrt{t})$, compared with the classical LT $\mathcal{F} = \mathcal{F}_{2,1}$ of the same function.

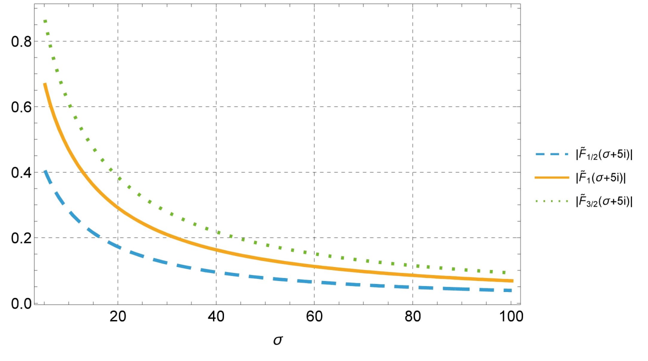


**Fig. 5.** Comparing the fractional LTs $\mathcal{F}_{2,1/2}$, $\mathcal{F}_{2,1}, \mathcal{F}_{2,3/2}$, of the function $J_0(2\sqrt{t})$ - the case of the modulus, assuming $s = \sigma + 5i$

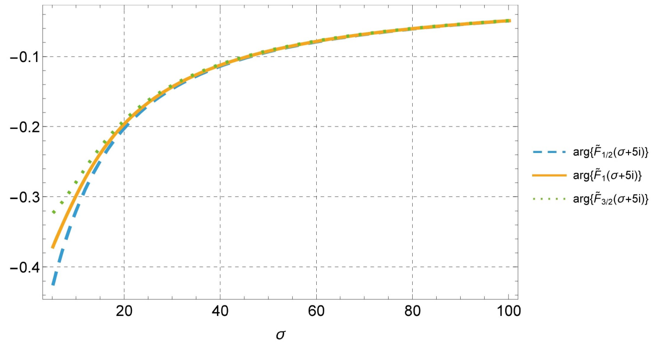


**Fig. 6.** Comparing the fractional LTs $\mathcal{F}_{2,1/2}$, $\mathcal{F}_{2,1}, \mathcal{F}_{2,3/2}$, of the function $J_0(2\sqrt{t})$ - the case of the argument, assuming $s = \sigma + 5i$

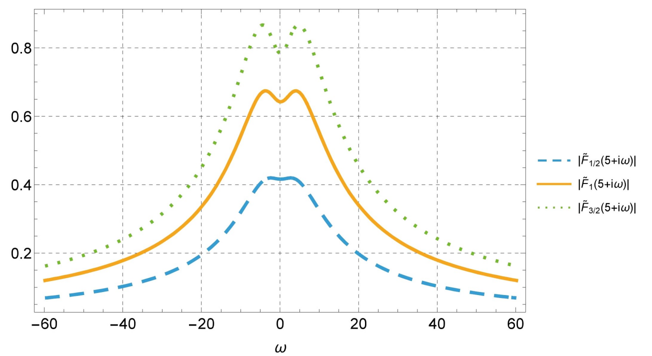


**Fig. 7.** Comparing the fractional LTs $\mathcal{F}_{2,1/2}$, $\mathcal{F}_{2,1}, \mathcal{F}_{2,3/2}$, of the function $J_0(2\sqrt{t})$ - the case of the modulus, assuming $s = 5 + i\,\omega$

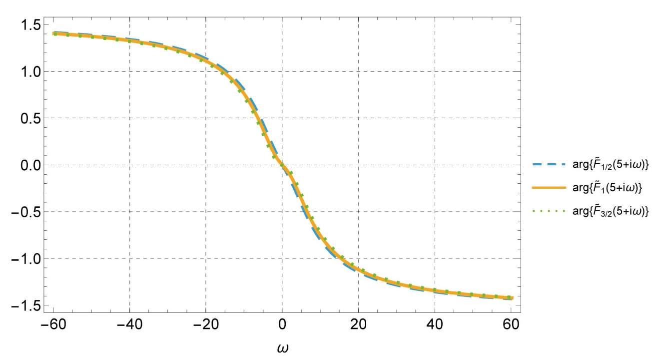


**Fig. 8.** Comparing the fractional LTs $\mathcal{F}_{2,1/2}$, $\mathcal{F}_{2,1}, \mathcal{F}_{2,3/2}$, of the function $J_0(2\sqrt{t})$ - the case of the argument, assuming $s = 5 + i\,\omega$

### 4.3. Example 2. (m=0)

Assuming $m = 0$, consider the fractional Laplace Transforms $\mathcal{F}_{1/2} = \mathcal{F}_{0,1/2}$, $\mathcal{F}_{3/2} = \mathcal{F}_{0,3/2}$ of the function $\exp(i\pi t)$, compared with the classical LT $\mathcal{F} = \mathcal{F}_{0,1}$ of the same function.

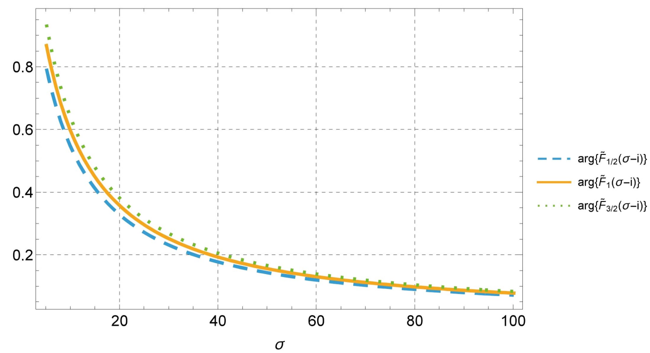


**Fig. 9.** Comparing the fractional LTs $\mathcal{F}_{0,1/2}$, $\mathcal{F}_{0,1}, \mathcal{F}_{0,3/2}$, of the function $\exp(i\pi t)$ - the case of the modulus, assuming $s = \sigma - i$

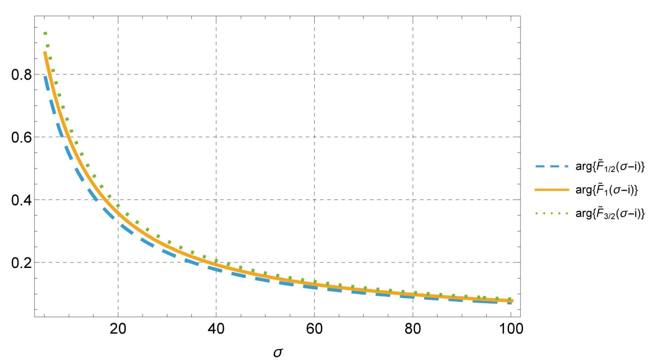


**Fig. 10.** Comparing the fractional LTs $\mathcal{F}_{0,1/2}$, $\mathcal{F}_{0,1}, \mathcal{F}_{0,3/2}$, of the function $\exp(i\pi t)$ - the case of the argument, assuming $s = \sigma - i$

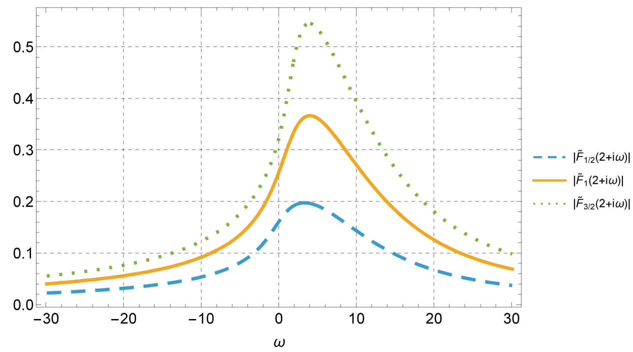


**Fig. 11.** Comparing the fractional LTs $\mathcal{F}_{0,1/2}$, $\mathcal{F}_{0,1}, \mathcal{F}_{0,3/2}$, of the function $\exp(i\pi t)$ - the case of the modulus, assuming $s = 2 + i\,\omega$

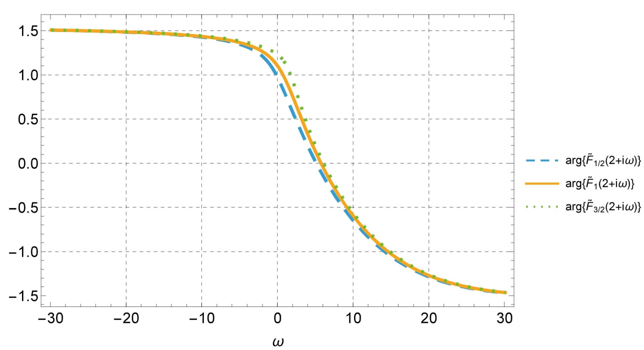


**Fig. 12.** Comparing the fractional LTs $\mathcal{F}_{0,1/2}$, $\mathcal{F}_{0,1}, \mathcal{F}_{0,3/2}$, of the function $\exp(i\pi t)$ - the case of the argument, assuming $s = 2 + i\,\omega$

## 4.4. Example 2. (m=2)

Assuming $m = 2$, consider the fractional Laplace Transforms $\mathcal{F}_{1/2} = \mathcal{F}_{2,1/2}$, $\mathcal{F}_{3/2} = \mathcal{F}_{2,3/2}$ of the function $\exp(i\pi t)$, compared with the classical LT $\mathcal{F} = \mathcal{F}_{2,1}$ of the same function.

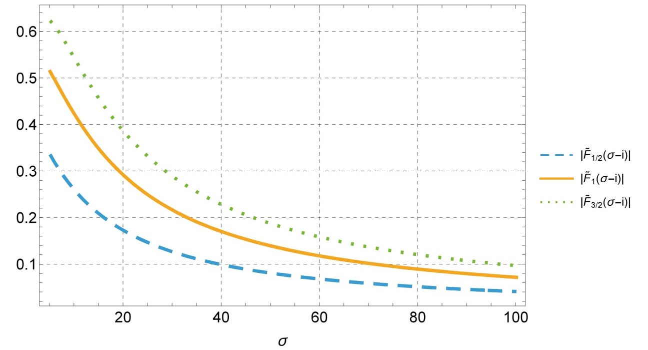


**Fig. 13.** Comparing the fractional LTs $\mathcal{F}_{2,1/2}$, $\mathcal{F}_{2,1}, \mathcal{F}_{2,3/2}$, of the function $\exp(i\pi t)$ - the case of the modulus, assuming $s = \sigma - i$

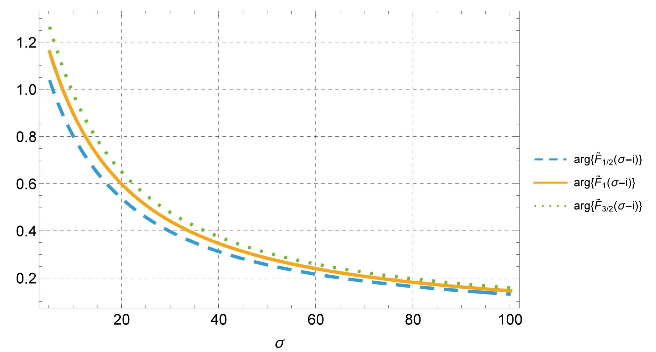


**Fig. 14.** Comparing the fractional LTs $\mathcal{F}_{2,1/2}$, $\mathcal{F}_{2,1}, \mathcal{F}_{2,3/2}$, of the function $\exp(i\pi t)$ - the case of the argument, assuming $s = \sigma - i$

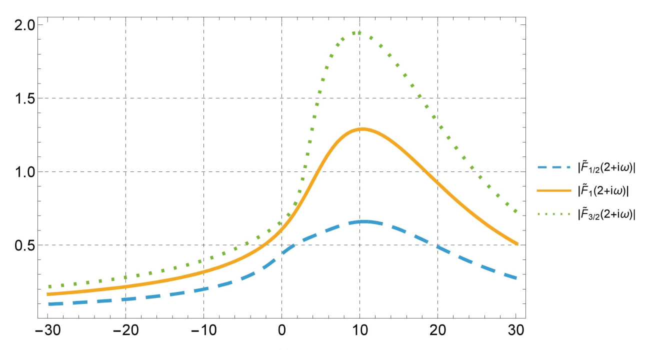


**Fig. 15.** Comparing the fractional LTs $\mathcal{F}_{2,1/2}$, $\mathcal{F}_{2,1}, \mathcal{F}_{2,3/2}$, of the function $\exp(i\pi t)$ - the case of the modulus, assuming $s = 2 + i\,\omega$

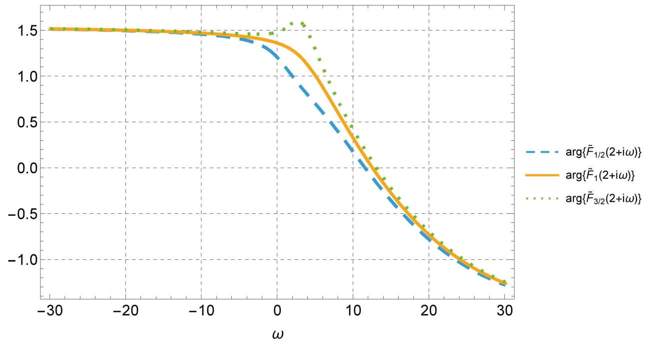


**Fig. 16.** Comparing the fractional LTs $\mathcal{F}_{2,1/2}$, $\mathcal{F}_{2,1}, \mathcal{F}_{2,3/2}$, of the function $\exp(i\pi t)$ - the case of the argument, assuming $s = 2 + i\,\omega$

## 4.5. Example 3. (m=0)

Assuming $m = 0$, consider the fractional Laplace Transforms $\mathcal{F}_{1/2} = \mathcal{F}_{0,1/2}$, $\mathcal{F}_{3/2} = \mathcal{F}_{0,3/2}$ of the function $\exp(-\sqrt{t})/\sqrt{t}$, compared with the classical LT $\mathcal{F} = \mathcal{F}_{0,1}$ of the same function.

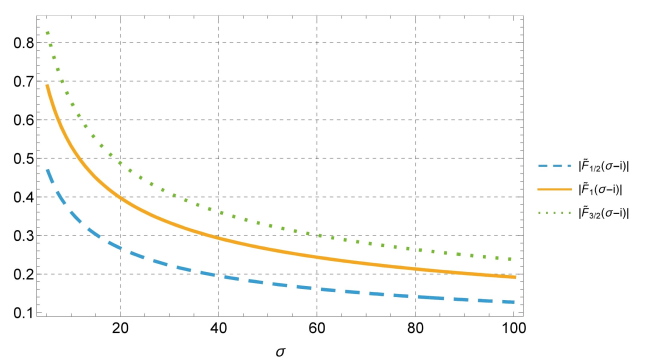


**Fig. 17.** Comparing the fractional LTs $\mathcal{F}_{0,1/2}$, $\mathcal{F}_{0,1}, \mathcal{F}_{0,3/2}$, of the function $\exp(-\sqrt{t})/\sqrt{t}$ - the case of the modulus, assuming $s = \sigma - i$

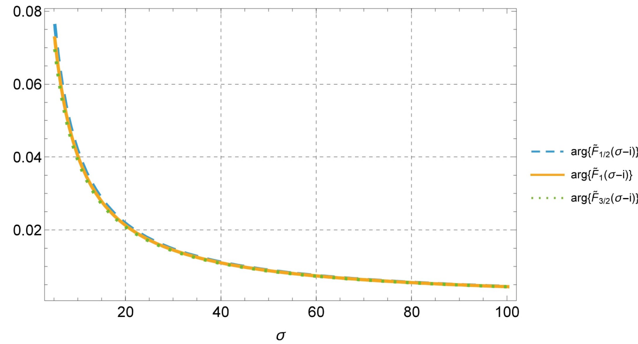


**Fig. 18.** Comparing the fractional LTs $\mathcal{F}_{0,1/2}$, $\mathcal{F}_{0,1}, \mathcal{F}_{0,3/2}$, of the function $\exp(-\sqrt{t})/\sqrt{t}$ - the case of the argument, assuming $s = \sigma - i$

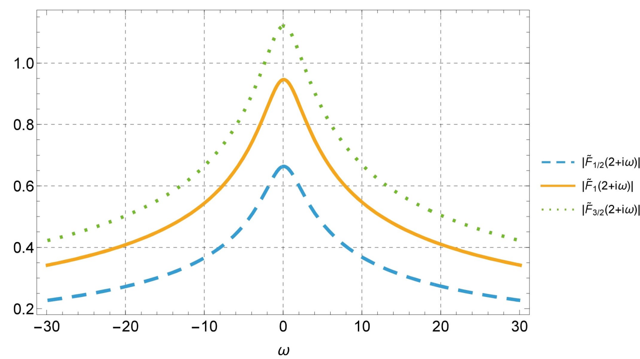


**Fig. 19.** Comparing the fractional LTs $\mathcal{F}_{0,1/2}$, $\mathcal{F}_{0,1}, \mathcal{F}_{0,3/2}$, of the function $\exp(-\sqrt{t})/\sqrt{t}$ - the case of the modulus, assuming $s = 2 + i\,\omega$

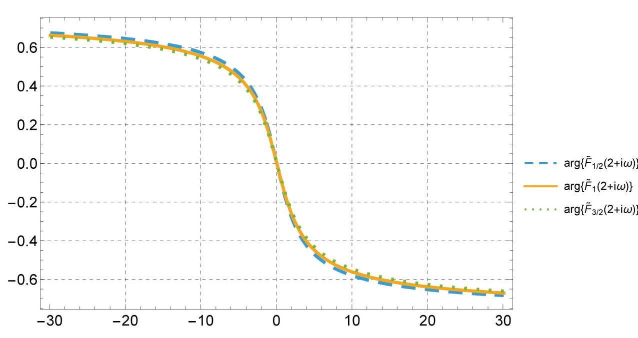


**Fig. 20.** Comparing the fractional LTs $\mathcal{F}_{0,1/2}$, $\mathcal{F}_{0,1}, \mathcal{F}_{0,3/2}$, of the function $\exp(-\sqrt{t})/\sqrt{t}$ - the case of the argument, assuming $s = 2 + i\,\omega$

## 4.6. Example 3. (m=2)

Assuming $m = 2$, consider the fractional Laplace Transforms $\mathcal{F}_{1/2} = \mathcal{F}_{2,1/2}$, $\mathcal{F}_{3/2} = \mathcal{F}_{2,3/2}$ of the function $\exp(-\sqrt{t})/\sqrt{t}$, compared with the classical LT $\mathcal{F} = \mathcal{F}_{2,1}$ of the same function.

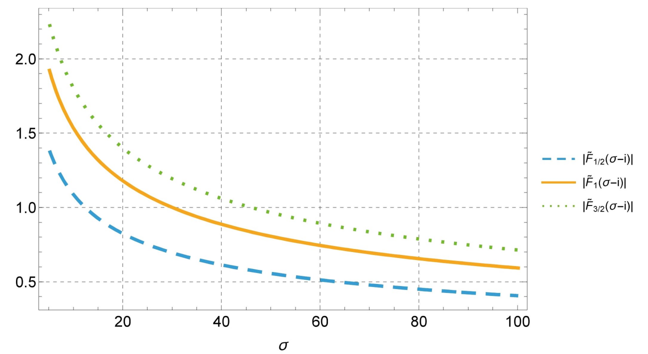


**Fig. 21.** Comparing the fractional LTs $\mathcal{F}_{2,1/2}$, $\mathcal{F}_{2,1}, \mathcal{F}_{2,3/2}$, of the function $\exp(-\sqrt{t})/\sqrt{t}$ - the case of the modulus, assuming $s = \sigma - i$

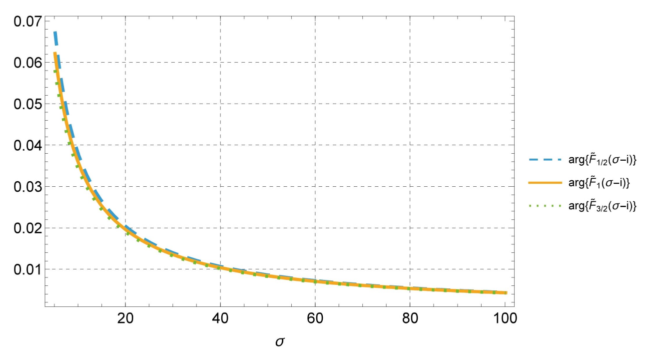


**Fig. 22.** Comparing the fractional LTs $\mathcal{F}_{2,1/2}$, $\mathcal{F}_{2,1}, \mathcal{F}_{2,3/2}$, of the function $\exp(-\sqrt{t})/\sqrt{t}$ - the case of the argument, assuming $s = \sigma - i$

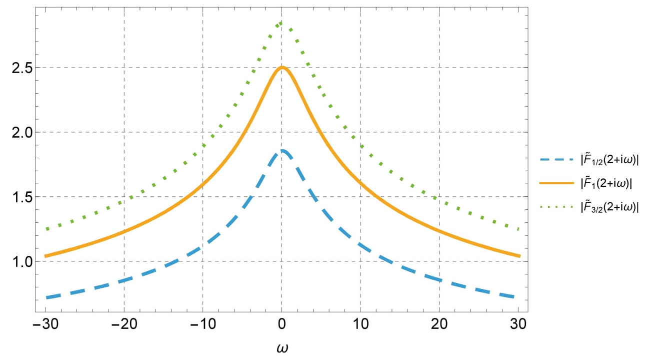


**Fig. 23.** Comparing the fractional LTs $\mathcal{F}_{2,1/2}$, $\mathcal{F}_{2,1}, \mathcal{F}_{2,3/2}$, of the function $\exp(-\sqrt{t})/\sqrt{t}$ - the case of the modulus, assuming $s = 2 + i\,\omega$

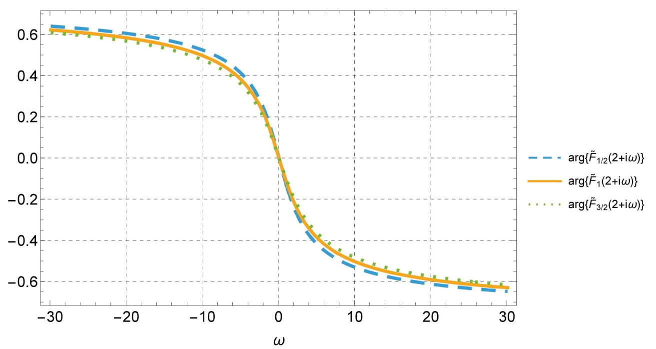


**Fig. 24.** Comparing the fractional LTs $\mathcal{F}_{2,1/2}$, $\mathcal{F}_{2,1}, \mathcal{F}_{2,3/2}$, of the function $\exp(-sqrtt/t)$ - the case of the argument, assuming $s = 2 + i\,\omega$

**Remark 3.** Many other tests have been performed, by the second author, using the same procedure, including the functions $e^t\,\Gamma(t)$, $\exp(t^2)$, $\mathrm{Sinc}(t)$, $J_1(t)/t$, $\cos(t^2)$, setting the parameter $m = 0, 1, 2$. The relevant graphs are available at his email address.

## 5. Conclusion

We have shown that, using the parametric Laguerre-type exponentials and their fractional versions, it is possible to define Laguerre-type parameric forms of the classical Laplace transform. We have used a general result to construct the reciprocals of some exponential-type functions, and we have used these reciprocals in place of the kernel of the usual Laplace transform.
Several worked examples of the new transformations, computed using the computer algebra system Mathematica© have been reported in the preceding Sections.

The introduced transformations could be used in the framework of fractional differential equations or in that of the Laguerre-type ones.

## Declarations

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