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Research Paper

A Probabilistic Approach to Frobenius-Genocchi Polynomials Derived From Polyexponential Function Associated With Their Certain Applications

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Abstract

In this paper, we introduce probabilistic extensions of polyexponential-Frobenius-Genocchi polynomials. By making use of their generating functions, we derive new and interesting identities among aforementioned polynomials and probabilistic Frobenius-Euler polynomials, the Stirling numbers of the first and second kind, probabilistic Frobenius-Genocchi polynomials, Bernoulli polynomials of the second kinds, Daehee polynomials, polyexponential probabilistic Bernoulli and polyexponential probabilistic Genocchi polynomials. In special cases, the obtained results reduce to the classical ones. Additionally, by picking suitable random variables, we also obtain new relations involving the Stirling numbers of the first and second kinds, and the Frobenius-Euler numbers.

Key Words: Moment generating function, Probabilisitic Frobenius-Genocchi polynomials, Polyexponential-Frobenius-Genocchi polynomials, Random variables.

AMS 2020 Classification: 11B73, 11B83, 05D40.

1. Introduction

Special functions and polynomials commonly arise as solutions to both ordinary and partial differential equations. Their significant importance are widely used in areas such as approximation theory, probability and statistics theory, analytic number theory, p-adic analysis, quantum analysis and numerical analysis. Many researchers have investigated the various properties and relations of special functions and polynomials.

In [1], Kim and Kim introduced polyexponential functions as an inverse to the polylogarithm functions, and constructed type 2 poly-Bernoulli polynomials, and then they derived various properties of type 2 poly-Bernoulli numbers. In [2], Araci introduced partially degenerate polyexponential-Bernoulli polynomials of the second kind. He also derived some identities for these polynomials including type 2-Euler polynomials and Stirling numbers of the first kind via generating function techniques and analytical means, and represented Gaussian integral representation of polyexponential-Bernoulli polynomials of the second kind. In [3], Khan *et al.* studied the degenerate polyexponential-Genocchi polynomials and numbers arising from the polyexponential function and derived their explicit expressions and some identities involving them. In [4], Corcino *et al.* introduced a class of polynomials called type 2 poly-Frobenius-Euler polynomials, and defined using the polyexponential function, and explored explicit expressions and identities for these polynomials. In [5], Duran *et al.* considered and investigated a new class of the Frobenius–Genocchi polynomials associated with its identities by means of the polyexponential function. In [6], Kim *et al.* studied the degenerate poly-Bernoulli polynomials and numbers arising from polyexponential functions, and derived their explicit expressions and some identities involving them. In [8], Lee introduced degenerate poly r-Stirling numbers of the second kind and poly r-Bell polynomials by using degenerate polyexponential function and investigate some properties of these numbers and polynomials. In [9], Araci considered a new class of generating functions of type 2-Bernoulli polynomials. He gave some identities for these polynomials including type 2-Euler polynomials and Stirling numbers of the second kind. In [11], Ma and Lim also studied the poly-Bernoulli numbers of the second kind, which are defined by using polyexponential functions introduced by Kim and Kim [1]. In the year 2019, Adell and Leukona [12] introduced a new generalization of the Stirling numbers of the second kind, associated with each complex-valued random variable satisfying appropriate integrability conditions. Then, in [13], Adell produced the applications for probabilistic Stirling numbers of the second kind associated with Y. Motivated by the aforementioned Adell's works, in [14], Kim and Kim studied probabilistic Bernoulli polynomials of order r associated with Y and probabilistic multi-poly-Bernoulli polynomials associated with Y. They had respectively probabilistic extensions of Bernoulli polynomials of order r and multi-poly-Bernoulli polynomials. They also found explicit expressions, certain related identities and some properties for them and treat the special cases of Poisson, Gamma and Bernoulli random variables. In [15], Karagenc *et al.* introduced probabilistic Bernstein polynomials and have derived new and interesting correlations among various special functions and special number sequences, such as Euler polynomials, higher-order Bernoulli polynomials, higher-order Frobenius-Euler polynomials, Stirling numbers of the second kind, and Bell polynomials. In [18], Karagenc *et al.* introduced probabilistic extensions of poly-Frobenius-Genocchi polynomials and modified probabilistic Genocchi-polynomials. By making use of their generating functions, they derived explicit identities and a symmetric relation. For more information on probabilistic special functions and polynomials, we refer the readers to look at [7, 17, 20].

We now begin with the following notations:

$$\mathbb{N} := \{1, 2, 3, ...\}, \mathbb{N}_0 := \{0, 1, 2, 3, ...\} = \mathbb{N} \cup \{0\}.$$

Also, as usual $\mathbb{Z}, \mathbb{Q}, \mathbb{R}^+$ will, respectively, be denoted integers, rational numbers, and positive real numbers. Let Y be chosen as a random variable satisfying the moment conditions by

$$\mathrm{E}\left[|Y|^n\right] < \infty \text{ and } \lim_{n\to\infty} \frac{|t|^n \mathrm{E}\left[|Y|^n\right]}{n!} = 0, (|t| < r; r > 0)$$

where E means mathematical expectation. From here, one may write that

$$\mathrm{E}\left[e^{tY}\right] = \sum_{n=0}^{\infty} \mathrm{E}\left[Y^n\right]^n \frac{t^n}{n!}, \left(|t| < r, r \in \mathbb{R}^+\right)$$

or equivalently,

$$\mathrm{E}\left[|Y|^n\right] < \infty, \ \left(|t| < r, r \in \mathbb{R}^+\right).$$

$\{Y_j\}_{j=1}^{k}$ is a sequence of mutually independent copies of the Y with $S_k = Y_1 + Y_2 + ... + Y_k$, $(k \in \mathbb{N})$ with $S_0 = 0$. (see [13], [15]).

Several distributions that will be used in deriving the results of this paper are given below:

1. ***Poisson distribution.*** $Y \sim Poisson(\alpha)$ with the parameter α yielding moment generating function (mgf) as

$$\mathrm{E}\left[e^{tY}\right] = e^{\alpha\left(e^t - 1\right)}. \tag{1}$$

2. ***Gamma distribution.*** Let $Y \sim \Gamma(1,1)$ be random variable with mgf as

$$\mathrm{E}\left[e^{tY}\right] = \frac{1}{1-t},\ t<1. \tag{2}$$

3. ***Exponential distribution.*** Let $Y \sim E(\alpha)$ be random variable with mgf as

$$\mathrm{E}\left[e^{tY}\right] = \frac{1}{1-\alpha t},\ 0<\alpha;\ t<\frac{1}{\alpha}.$$

(see [13], [15]).

We now reminder polyexponential functions $\mathrm{Ei}_k(t)$, Frobenius-Genocchi polynomials $G_n^F(x,u)$, Frobenius-Euler polynomials $H_n(x,u)$, Bernoulli polynomials of the second kind $b_n(x)$, Stirling numbers of the first kind $S_1(n,k)$, Stirling numbers of the second kind $S_2(n,k)$ and Daehee polynomials $D_n(x)$. Also we recall the probabilistic Frobenius-Euler polynomials $H_n^Y(x,u)$, probabilistic Frobenius-Genocchi polynomials $G_n^{(F,Y)}(x,u)$.

In [1], [6], it is well known that the polyexponential function is defined by

$$\mathrm{Ei}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!n^k}, (k \in \mathbb{Z}),$$

for $k=1$, $\mathrm{Ei}_1(x) = e^x - 1$, and has the following derivative property:

$$\frac{d}{dt}\mathrm{Ei}_k(f(t)) = \frac{f'(t)}{f(t)}\mathrm{Ei}_{k-1}(f(t)).$$

In [10], [16], Frobenius-Euler polynomials are defined as

$$\frac{1-u}{e^t-u}e^{xt} = \sum_{n=0}^{\infty} H_n(x,u)\frac{t^n}{n!}, (u \in \mathbb{C}-\{1\}).$$

In [5], [18], Frobenius-Genocchi polynomials $G_n^F(x,u)$ are given by

$$\frac{(1-u)t}{e^t-u}e^{xt} = \sum_{n=0}^{\infty} G_n^F(x,u)\frac{t^n}{n!}, (u \in \mathbb{C}-\{1\}),$$

and

$$\frac{G_{n+1}^F(x,u)}{n+1} = H_n(x,u).$$

In [12], the Stirling numbers of the first kind, $S_1(n,k)$, are defined by means of the following generating function:

$$\frac{(\log(1+t))^k}{k!} = \sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!}.$$

In [12], the Stirling numbers of the second kind, $S_1(n,k)$, are defined by means of the following generating function:

$$\frac{(e^t-1)^k}{k!} = \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!}.$$

In [19], Daehee polynomials, $D_n(x)$, are defined by means of the following generating function:

$$\frac{\log(1+t)}{t}(1+t)^x = \sum_{n=0}^{\infty} D_n(x)\frac{t^n}{n!}.$$

In [9], Bernoulli polynomials of the second kind, (or the Cauchy polynomials) are given by the generating function to be

$$\frac{t}{\log(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!}.$$

Throughout of this paper, we will assume as follows: If $1+t$ becomes a complex number, the value of $\log(1+t)$ is then defined by

$$\log(1+t) = \log|1+t| + i\arg(1+t), \tag{3}$$

where

1. $\log|1+t|$ is the real-valued logarithm of the modulus $|1+t|$.
2. $\arg(1+t)$ is the angle of $1+t$, typically restricted to the principal branch $(-\pi, \pi]$.

In [17], probabilistic Frobenius-Euler polynomials and probabilistic Frobenius-Genocchi polynomials subject to a random variable Y are known as, respectively:

$$\frac{1-u}{\mathrm{E}\left[e^{tY}\right]-u}\left(\mathrm{E}\left[e^{tY}\right]\right)^x = \sum_{n=0}^{\infty} H_n^Y(x,u)\frac{t^n}{n!}$$

and

$$\frac{(1-u)t}{\mathrm{E}\left[e^{tY}\right]-u}\left(\mathrm{E}\left[e^{tY}\right]\right)^x = \sum_{n=0}^{\infty} G_n^{(F,Y)}(x,u)\frac{t^n}{n!},$$

where $(u \in \mathbb{C} - \{1\})$. In the case when $Y = 1$, $H_n^Y(x) = H_n(x)$ and $G_n^{(F,Y)}(x) = G_n(x)$ turn out to be well known (classical or ordinary) Frobenius-Euler and Frobenius-Genocchi polynomials. Also, at the value of $x = 0$, $H_n^Y(0) = H_n^Y$ and $G_n^{(F,Y)}(0) = G_n^Y$ are called the probabilistic Frobenius-Euler numbers and probabilistic Frobenius-Genocchi numbers.
The probabilistic polyexponential Bernoulli polynomials and probabilistic polyexponential Genocchi polynomials subject to a random variable Y can be expressed as, respectively:

$$\sum_{n=0}^{\infty} \underline{B}_n^{(k,Y)}(x)\frac{t^n}{n!} = \frac{\mathrm{Ei}_k(\log(1+t))}{\mathrm{E}[e^{tY}]-1}(\mathrm{E}[e^{tY}])^x, \tag{4}$$

and

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(k,Y)}(x)\frac{t^n}{n!} = \frac{2\mathrm{Ei}_k(\log(1+t))}{\mathrm{E}[e^{tY}]+1}(\mathrm{E}[e^{tY}])^x.$$

In the next section, we introduce the generating function for the probabilistic polyexponential-Frobenius-Genocchi polynomials. Utilizing this generating function, we derive explicit identities. Furthermore, by selecting appropriate random variables, we establish connections between the aforementioned polynomial and other special functions and polynomials.

2. Introducing Probabilistic Polyexponential-Frobenius-Genocchi Polynomials With Their Certain Identities

We are now in a position to state the following theorem, which serves as the main definition of this paper for deriving new identities, relations, and properties.

Definition 1. Let Y be a random variable and $k \in \mathbb{Z}, u \in \mathbb{C}, u \neq 1$. The probabilistic polyexponential-Frobenius-Genocchi polynomials $\underline{G}_n^{(k,Y)}(x,u)$ associated with Y are defined by

$$\sum_{n=0}^{\infty} \underline{G}_n^{(k,Y)}(x,u)\frac{t^n}{n!} = \frac{(1-u)\mathrm{Ei}_k(\log(1+t))}{\mathrm{E}[e^{tY}]-u}(\mathrm{E}[e^{tY}])^x. \tag{5}$$

For $x = 0, \underline{G}_n^{(k,Y)}(0,u) := \underline{G}_n^{(k,Y)}(u)$ are called probabilistic polyexponential-Frobenius-Genocchi numbers.

Remark 1. In the case when, $k=1$, $\underline{G}_n^{(1,Y)}(x,u) := \underline{G}_n^{(F,Y)}(x,u)$ turns out to be probabilistic Frobenius-Genocchi polynomials.

Remark 2. In the case when, $Y=1$, $\underline{G}_n^{(k,1)}(x,u) := \underline{G}_n^{k}(x,u)$ gives the polyexponential-Frobenius-Genocchi polynomials.

Remark 3. In the case when, $Y=1, k=1$, $\underline{G}_n^{(1,1)}(x,u) := \underline{G}_n(x,u)$ becomes Frobenius-Genocchi polynomials.

Theorem 1. *The probabilistic polyexponential-Frobenius-Genocchi numbers associated with Y can be expressed as*

$$\underline{G}_{n+1}^{(k,Y)}(x,u) = \sum_{i=0}^{n}\sum_{m=0}^{i}\binom{n+1}{i+1}H_{n-i}^{Y}(x,u)\frac{S_1(i+1,m+1)}{(m+1)^{k-1}}.$$

Proof. Since

$$\begin{aligned}
\sum_{n=0}^{\infty}\underline{G}_n^{(k,Y)}(x,u)\frac{t^n}{n!} &= \frac{(1-u)\mathrm{Ei}_k(\log(1+t))}{\mathrm{E}[e^{tY}]-u}(\mathrm{E}[e^{tY}])^x \\
&= \frac{(1-u)(\mathrm{E}[e^{tY}])^x}{\mathrm{E}[e^{tY}]-u}\sum_{m=1}^{\infty}\frac{(\log(1+t))^m}{(m-1)!m^k} \\
&= \sum_{n=0}^{\infty}H_n^Y(x,u)\frac{t^n}{n!}\sum_{m=0}^{\infty}\frac{(\log(1+t))^{m+1}}{(m+1)!(m+1)^{k-1}} \\
&= \sum_{n=0}^{\infty}H_n^Y(x,u)\frac{t^n}{n!}\sum_{m=0}^{\infty}\frac{1}{(m+1)^{k-1}}\sum_{n=m+1}^{\infty}S_1(n,m+1)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}H_n^Y(x,u)\frac{t^n}{n!}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{S_1(n,m+1)}{(m+1)^{k-1}}\frac{t^{(n+1)}}{(n+1)!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\sum_{m=0}^{i}\binom{n+1}{i+1}H_{n-i}^{Y}(x,u)\frac{S_1(i+1,m+1)}{(m+1)^{k-1}}\right)\frac{t^{n+1}}{(n+1)!}.
\end{aligned}$$

Thus, by comparing the coefficients $\frac{t^n}{n!}$ on both sides of the above, we arrive at the desired result. □

Theorem 2. *Let Y be a random variable. Then we have the explicit identity:*

$$\sum_{m=0}^{n+1}\underline{G}_m^{(k,Y)}(u)S_2(n+1,m) = \sum_{i=0}^{n}\sum_{m=0}^{n-i}\binom{n+1}{i+1}\frac{S_2(n-i,m)H_m^Y(u)}{(i+1)^{k-1}}.$$

Proof. Since

$$\sum_{n=0}^{\infty}\underline{G}_n^{(k,Y)}(u)\frac{t^n}{n!} = \frac{(1-u)\mathrm{Ei}_k(\log(1+t))}{\mathrm{E}[e^{tY}]-u},$$

and replacing t by e^t-1 in (5) in this case, we obtain

$$\sum_{m=0}^{\infty}\underline{G}_m^{(k,Y)}(u)\frac{(e^t-1)^m}{m!} = \sum_{m=0}^{\infty}\underline{G}_m^{(k,Y)}(u)\sum_{n=m}^{\infty}S_2(n,m)\frac{t^m}{m!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \underline{G}_m^{(k,Y)}(u) S_2(n,m) \right) \frac{t^n}{n!}.$$

Then, we get

$$\begin{aligned}
\frac{(1-u)\mathrm{Ei}_k(\log(1+e^t-1))}{\mathrm{E}[e^{(e^t-1)Y}]-u} &= \frac{(1-u)\mathrm{Ei}_k(t)}{\mathrm{E}[e^{(e^t-1)Y}]-u} \\
&= \frac{(1-u)}{\mathrm{E}[e^{(e^t-1)Y}]-u} \sum_{n=0}^{\infty} \frac{t^{n+1}}{(n+1)!(n+1)^{k-1}} \\
&= \left(\sum_{m=0}^{\infty} H_m^Y(u) \frac{(e^t-1)^m}{m!} \right) \left(\sum_{n=0}^{\infty} \frac{t^{n+1}}{(n+1)!(n+1)^{k-1}} \right) \\
&= \left(\sum_{m=0}^{\infty} H_m^Y(u) \sum_{n=m}^{\infty} S_2(n,m) \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} \frac{t^{n+1}}{(n+1)!(n+1)^{k-1}} \right) \\
&= \left(\sum_{n=0}^{\infty} \sum_{m=0}^{n} S_2(n,m) H_m^Y(u) \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} \frac{1}{(n+1)^{k-1}} \frac{t^{n+1}}{(n+1)!} \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} \sum_{m=0}^{n-i} \binom{n+1}{i+1} \frac{S_2(n-i,m) H_m^Y(u)}{(i+1)^{k-1}} \right) \frac{t^{n+1}}{(n+1)!}.
\end{aligned}$$

Therefore, equating the coefficients of $\frac{t^n}{n!}$ on both sides yields the desired result. □

Theorem 3. *Let Y be a random variable. Then the following equality holds true:*

$$\underline{G}_n^{(1,Y)}(x,u) = G_n^{(F,Y)}(x,u).$$

Proof. By (5) for $k=1$, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \underline{G}_n^{(1,Y)}(x,u) \frac{t^n}{n!} &= \frac{(1-u)\mathrm{Ei}_1(\log(1+t))}{\mathrm{E}[e^{tY}]-u} (\mathrm{E}[e^{tY}])^x \\
&= \frac{(1-u)t}{\mathrm{E}[e^{tY}]-u} (\mathrm{E}[e^{tY}])^x \\
&= \sum_{n=0}^{\infty} G_n^{(F,Y)}(x,u) \frac{t^n}{n!}.
\end{aligned}$$

Thus by comparing coefficients $\frac{t^n}{n!}$ on both sides of the above, we arrive at the desired result.
□

Let us now consider the following expression:

$$\begin{aligned}
\frac{d}{dx}\mathrm{Ei}_k(\log(1+x)) &= \frac{d}{dx} \sum_{n=1}^{\infty} \frac{(\log(1+x))^n}{(n+1)!n^k} \\
&= \frac{1}{(1+x)\log(1+x)} \sum_{n=1}^{\infty} \frac{(\log(1+x))^n}{(n+1)!n^(k-1)} \qquad (6) \\
&= \frac{1}{(1+x)\log(1+x)} \mathrm{Ei}_{k-1}(\log(1+x)),
\end{aligned}$$

and we easily derive that

$$\begin{aligned}\mathrm{Ei}_k(\log(1+x)) &= \int_0^x \frac{1}{(1+t_1)\log(1+t_1)}\mathrm{Ei}_{k-1}(\log(1+t_1))dt_1\\ &= \int_0^x \frac{1}{(1+t_1)\log(1+t_1)}\int_0^{t_1}\frac{1}{(1+t_2)\log(1+t_2)}\mathrm{Ei}_{k-2}(\log(1+t_2))dt_2dt_1\\ &= \int_0^x \frac{1}{(1+t_1)\log(1+t_1)}\int_0^{t_1}\frac{1}{(1+t_2)\log(1+t_2)}\cdots\int_0^{t_{k-2}}\frac{t_{k-1}}{(1+t_{k-1})\log(1+t_{k-1})}\prod_{i=1}^{k-1}dt_i.\end{aligned} \tag{7}$$

From (7), we state the following theorem.

Theorem 4. *Let Y be a random variable, then we have the explicit identity:*

$$\underline{G}_{n+1}^{(2,Y)}(x,u) = \sum_{i=0}^{n}\binom{n+1}{i+1}b_i(-1)H_{n-i}^{Y}(x,u).$$

Proof. By (5) for $k=2$, (6) and (7), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\underline{G}_n^{(2,Y)}(x,u)\frac{t^n}{n!} &= \frac{(1-u)(\mathrm{E}[e^{tY}])^x}{\mathrm{E}[e^{tY}]-u}\int_0^t \frac{d}{dy}\mathrm{Ei}_2(\log(1+y))dy\\ &= \sum_{n=0}^{\infty}H_n^Y(x,u)\frac{t^n}{n!}\int_0^t \frac{1}{(1+y)\log(1+y)}\mathrm{Ei}_1(\log(1+y))dy\\ &= \sum_{n=0}^{\infty}H_n^Y(x,u)\frac{t^n}{n!}\int_0^t \frac{y}{(1+y)\log(1+y)}dy\\ &= \sum_{n=0}^{\infty}H_n^Y(x,u)\frac{t^n}{n!}\sum_{n=0}^{\infty}\frac{b_n(-1)}{n!}\int_0^t y^n dy\\ &= \sum_{n=0}^{\infty}H_n^Y(x,u)\frac{t^n}{n!}\sum_{n=0}^{\infty}b_n(-1)\frac{t^{n+1}}{(n+1)!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\binom{n+1}{i+1}b_i(-1)H_{n-i}^Y(x,u)\right)\frac{t^{n+1}}{(n+1)!}.\end{aligned}$$

By comparing the coefficients of $\frac{t^n}{n!}$ on both sides, we obtain the desired identity. □

The following theorem forms of the product of probabilistic Frobenius-Euler polynomials and polyexponential probabilistic Bernoulli polynomials.

Theorem 5. *Let Y be a random variable, we get*

$$\underline{G}_n^{(k,Y)}(x,u) = \sum_{i=0}^{n}\binom{n}{i}\left(H_{n-i}^Y(1,u)-H_{n-i}^Y(u)\right)\underline{B}_i^{(k,Y)}(x).$$

Proof. Since

$$\begin{aligned}
\sum_{n=0}^{\infty} \underline{G}_n^{(k,Y)}(x,u)\frac{t^n}{n!} &= \frac{(1-u)\mathrm{Ei}_k(\log(1+t))}{\mathrm{E}[e^{tY}]-u}(\mathrm{E}[e^{tY}])^x \\
&= \frac{(1-u)(\mathrm{E}[e^{tY}]-1)}{\mathrm{E}[e^{tY}]-u}\frac{\mathrm{Ei}_k(\log(1+t))}{\mathrm{E}[e^{tY}]-1}(\mathrm{E}[e^{tY}])^x \\
&= \frac{(1-u)(\mathrm{E}[e^{tY}]-1)}{\mathrm{E}[e^{tY}]-u}\sum_{n=0}^{\infty}\underline{B}_n^{(k,Y)}(x)\frac{t^n}{n!} \\
&= \frac{(1-u)\mathrm{E}[e^{tY}]}{\mathrm{E}[e^{tY}]-u}\sum_{n=0}^{\infty}\underline{B}_n^{(k,Y)}(x)\frac{t^n}{n!} - \frac{(1-u)}{\mathrm{E}[e^{tY}]-u}\sum_{n=0}^{\infty}\underline{B}_n^{(k,Y)}(x)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} H_n^Y(1,u)\frac{t^n}{n!}\sum_{n=0}^{\infty}\underline{B}_n^{(k,Y)}(x)\frac{t^n}{n!} - \sum_{n=0}^{\infty} H_n^Y(u)\frac{t^n}{n!}\sum_{n=0}^{\infty}\underline{B}_n^{(k,Y)}(x)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\binom{n}{i}\sum_{i=0}^{n} H_{n-i}^Y(1,u)\underline{B}_i^{(k,Y)}(x)\right)\frac{t^n}{n!} - \sum_{n=0}^{\infty}\left(\binom{n}{i}\sum_{i=0}^{n} H_{n-i}^Y(u)\underline{B}_i^{(k,Y)}(x)\right)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\binom{n}{i}\left(H_{n-i}^Y(1,u)-H_{n-i}^Y(u)\right)\underline{B}_i^{(k,Y)}(x)\right)\frac{t^n}{n!}.
\end{aligned}$$

Thus by comparing coefficients $\frac{t^n}{n!}$ on both sides of the above, we arrive at the desired result.
□

The following theorem includes the product of probabilistic Frobenius-Euler polynomials and probabilistic polyexponential Genocchi polynomials.

Theorem 6. *Let Y be a random variable, then we have the explicit identity:*

$$\underline{G}_n^{(k,Y)}(x,u) = \sum_{i=0}^{n}\binom{n}{i}\frac{\mathcal{G}_i^{(k,Y)}(x)}{2}\left(H_{n-i}^Y(1,u)+H_{n-i}^Y(u)\right).$$

Proof. We have

$$\begin{aligned}
\sum_{n=0}^{\infty} \underline{G}_n^{(k,Y)}(x,u)\frac{t^n}{n!} &= \frac{(1-u)\mathrm{Ei}_k(\log(1+t))}{\mathrm{E}[e^{tY}]-u}(\mathrm{E}[e^{tY}])^x \\
&= \frac{(1-u)(\mathrm{E}[e^{tY}]+1)}{2(\mathrm{E}[e^{tY}]-u)}\frac{2\mathrm{Ei}_k(\log(1+t))}{\mathrm{E}[e^{tY}]+1}(\mathrm{E}[e^{tY}])^x \\
&= \frac{(1-u)(\mathrm{E}[e^{tY}]+1)}{2(\mathrm{E}[e^{tY}]-u)}\sum_{n=0}^{\infty}\mathcal{G}_n^{(k,Y)}(x)\frac{t^n}{n!} \\
&= \frac{1}{2}\left(\frac{(1-u)\mathrm{E}[e^{tY}]}{\mathrm{E}[e^{tY}]-u}\right)\sum_{n=0}^{\infty}\mathcal{G}_n^{(k,Y)}(x)\frac{t^n}{n!} + \frac{1}{2}\left(\frac{(1-u)}{\mathrm{E}[e^{tY}]-u}\right)\sum_{n=0}^{\infty}\mathcal{G}_n^{(k,Y)}(x)\frac{t^n}{n!} \\
&= \left(\sum_{n=0}^{\infty} H_n^Y(1,u)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{\mathcal{G}_n^{(k,Y)}(x)}{2}\frac{t^n}{n!}\right) + \left(\sum_{n=0}^{\infty} H_n^Y(u)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{\mathcal{G}_n^{(k,Y)}(x)}{2}\frac{t^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\left(\binom{n}{i}\sum_{i=0}^{n} H_{n-i}^Y(1,u)\frac{\mathcal{G}_i^{(k,Y)}(x)}{2}\right)\frac{t^n}{n!} + \sum_{n=0}^{\infty}\left(\binom{n}{i}\sum_{i=0}^{n} H_{n-i}^Y(u)\frac{\mathcal{G}_i^{(k,Y)}(x)}{2}\right)\frac{t^n}{n!}
\end{aligned}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} \binom{n}{i} \frac{\mathcal{G}_i^{(k,Y)}(x)}{2} \left(H_{n-i}^{Y}(1,u) + H_{n-i}^{Y}(u) \right) \right) \frac{t^n}{n!}.$$

By comparing the coefficients of $\frac{t^n}{n!}$ on both sides, we obtain the desired identity. □

The following theorem forms of the triple summations for the products of probabilistic Frobenius-Euler polynomials, Stirling numbers of first kind and Daehee polynomials.

Theorem 7. *Let Y be a random variable, we get*

$$\frac{\underline{G}_{n+1}^{(k,Y)}(x,u)}{n+1} = \sum_{j=0}^{n} \sum_{i=0}^{j} \sum_{m=0}^{i} \binom{n}{j} \frac{H_{j-1}^{Y}(x,u) S_1(i,m) D_{n-j}}{(m+1)^k}.$$

Proof. Since

$$\begin{aligned}
\sum_{n=0}^{\infty} \underline{G}_n^{(k,Y)}(x,u) \frac{t^n}{n!} &= \frac{(1-u)}{\mathrm{E}[e^{tY}]-u} (\mathrm{E}[e^{tY}])^x \sum_{m=1}^{\infty} \frac{(\log(1+t))^m}{(m-1)! m^k} \\
&= \sum_{n=0}^{\infty} H_n^Y(x,u) \frac{t^n}{n!} \sum_{m=0}^{\infty} \frac{1}{(m+1)^k} \frac{(\log(1+t))^m}{m!} \frac{\log(1+t)}{t} t \\
&= \sum_{n=0}^{\infty} H_n^Y(x,u) \frac{t^n}{n!} \sum_{m=0}^{\infty} \frac{1}{(m+1)^k} \sum_{n=m}^{\infty} S_1(n,m) \frac{t^n}{n!} \sum_{n=0}^{\infty} D_n \frac{t^{n+1}}{n!} \\
&= \sum_{n=0}^{\infty} H_n^Y(x,u) \frac{t^n}{n!} \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \frac{1}{(m+1)^k} S_1(n,m) \right) \frac{t^n}{n!} \sum_{n=0}^{\infty} D_n \frac{t^{n+1}}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} \sum_{m=0}^{i} \binom{n}{i} \frac{1}{(m+1)^k} H_{n-i}^Y(x,u) S_1(i,m) \right) \frac{t^n}{n!} \sum_{n=0}^{\infty} (n+1) D_n \frac{t^{n+1}}{(n+1)!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} \sum_{i=0}^{j} \sum_{m=0}^{i} \binom{j}{i} \frac{n-j+1}{(m+1)^k} H_{j-i}^Y(x,u) S_1(i,m) D_{n-j} \right) \frac{t^j}{j!} \frac{t^{n-j+1}}{(n-j+1)!} \frac{(n+1)!}{(n+1)!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} \sum_{i=0}^{j} \sum_{m=0}^{i} \binom{n}{j} \frac{H_{j-i}^Y(x,u) S_1(i,m) D_{n-j}}{(m+1)^k} \right) \frac{t^{n+1}}{(n+1)!}.
\end{aligned}$$

Thus by comparing coefficients $\frac{t^n}{n!}$ on both sides of the above, we arrive at the desired result.
□

We first consider the following expression:

$$\begin{aligned}
\mathrm{Ei}_k(\log(1+t)) &= \sum_{m=1}^{\infty} \frac{(\log(1+t))^m}{(m-1)! m^k} \\
&= \sum_{m=1}^{\infty} \frac{1}{m^{k-1}} \sum_{n=m}^{\infty} S_1(n,m) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{m=1}^{n+1} \frac{S_1(n+1,m)}{m^{k-1}(n+1)} \right) \frac{t^{n+1}}{n!}.
\end{aligned}$$

Thus, we state the following theorem.

Theorem 8. *Let Y be a random variable, then we get*

$$\underline{G}_n^{(k,Y)}(x,u) = \sum_{i=0}^{n}\sum_{m=1}^{i+1}\binom{n}{i} G_{n-i}^{(F,Y)}(x,u)\frac{S_1(i+1,m)}{m^{k-1}(i+1)}.$$

Proof. We have

$$\begin{aligned}\sum_{n=0}^{\infty}\underline{G}_n^{(k,Y)}(x,u)\frac{t^n}{n!} &= \frac{(1-u)(\mathrm{E}[e^{tY}])^x}{\mathrm{E}[e^{tY}]-u}\mathrm{Ei}_k(\log(1+t)) \\ &= \frac{(1-u)t(\mathrm{E}[e^{tY}])^x}{\mathrm{E}[e^{tY}]-u}\sum_{n=0}^{\infty}\left(\sum_{m=1}^{n+1}\frac{S_1(n+1,m)}{m^{k-1}(n+1)}\right)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}G_n^{(F,Y)}(x,u)\frac{t^n}{n!}\sum_{n=0}^{\infty}\left(\sum_{m=1}^{n+1}\frac{S_1(n+1,m)}{m^{k-1}(n+1)}\right)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\sum_{m=1}^{i+1}\binom{n}{i}G_{n-i}^{(F,Y)}(x,u)\frac{S_1(i+1,m)}{m^{k-1}(i+1)}\right)\frac{t^n}{n!}.\end{aligned}$$

Thus by comparing coefficients $\frac{t^n}{n!}$ on both sides of the above, we arrive at the desired result.
□

By employing suitable random variables, we obtain new relations, formulae and identities involving the Stirling numbers of the first and second kinds, and the Frobenius-Euler numbers which we stated in the next section.

3. Applications

The following theorem consists of the product of Frobenius-Euler polynomials and Stirling numbers of first and second kind.

Theorem 9. *Let $Y \sim Poisson(\alpha)$, we get*

$$\underline{G}_{n+1}^{(k,Y)}(u) = \sum_{i=0}^{n}\sum_{j=0}^{n-i}\sum_{m=0}^{i}\binom{n+1}{i+1}\frac{H_j(u)}{(m+1)^{k+1}}\alpha^j S_2(n-i,j)S_1(i+1,m+1).$$

Proof. By (1) and (5) we have

$$\begin{aligned}\sum_{n=0}^{\infty}\underline{G}_n^{(k,Y)}(u)\frac{t^n}{n!} &= \frac{(1-u)}{e^{\alpha(e^t-1)}-u}\mathrm{Ei}_k(\log(1+t)) \\ &= \sum_{j=0}^{\infty}H_j(u)\alpha^j\frac{(e^t-1)^j}{j!}\sum_{m=0}^{\infty}\frac{(\log(1+t))^{m+1}}{(m+1)!(m+1)^{k-1}} \\ &= \left(\sum_{j=0}^{\infty}H_j(u)\alpha^j\sum_{n=j}^{\infty}S_2(n,j)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty}\frac{1}{(m+1)^{k+1}}\sum_{n=m+1}^{\infty}S_1(n,m+1)\frac{t^n}{n!}\right)\end{aligned}$$

$$= \left(\sum_{n=0}^{\infty} \sum_{j=0}^{n} H_j(u) \alpha^j S_2(n,j) \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{S_1(n+1,m+1)}{(m+1)^{k+1}} \frac{t^{n+1}}{(n+1)!} \right)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} \sum_{j=0}^{n-i} \sum_{m=0}^{i} \binom{n+1}{i+1} \frac{H_j(u)}{(m+1)^{k+1}} \alpha^j S_2(n-i,j) S_1(i+1,m+1) \right) \frac{t^{n+1}}{(n+1)!}.$$

By comparing coefficients $\frac{t^n}{n!}$ on both sides of the above, we arrive at the desired result.
□

The following theorem forms of the triple summations for the products of Frobenius-Euler polynomials and Stirling numbers of the first kind.

Theorem 10. *Let $Y \sim \Gamma(1,1)$, we get*

$$\underline{G}_{n+1}^{(k,Y)}(u) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} \sum_{m=1}^{i+1} \binom{n+1}{i+1} \frac{(-1)^{n-i+j}}{m^{k-1}} H_j(u) S_1(n-i,j) S_1(i+1,m).$$

Proof. By (2) and (5) we have

$$\sum_{n=0}^{\infty} \underline{G}_n^{(k,Y)}(x,u) \frac{t^n}{n!} = \frac{(1-u)}{\frac{1}{1-t} - u} \mathrm{Ei}_k(\log(1+t))$$

$$= \frac{(1-u)}{e^{\log(\frac{1}{1-t})} - u} \sum_{m=1}^{\infty} \frac{(\log(1+t))^m}{(m-1)! m^k}$$

$$= \sum_{j=0}^{\infty} H_j(u) (-1)^j \frac{(\log(1-t))^j}{j!} \sum_{m=1}^{\infty} \frac{1}{m^{k-1}} \sum_{n=m}^{\infty} S_1(n,m) \frac{t^n}{n!}$$

$$= \sum_{j=0}^{\infty} H_j(u) (-1)^j \sum_{n=j}^{\infty} (-1)^n S_1(n,j) \frac{t^n}{n!} \sum_{n=0}^{\infty} \left(\sum_{m=1}^{n+1} \frac{S_1(n+1,m)}{m^{k-1}} \right) \frac{t^{n+1}}{(n+1)!}.$$

$$= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} (-1)^{n+j} H_j(u) S_1(n,j) \right) \frac{t^n}{n!} \sum_{n=0}^{\infty} \left(\sum_{m=1}^{n+1} \frac{S_1(n+1,m)}{m^{k-1}} \right) \frac{t^{n+1}}{(n+1)!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} \sum_{j=0}^{n-i} \sum_{m=1}^{i+1} \binom{n+1}{i+1} \frac{(-1)^{n-i+j}}{m^{k-1}} H_j(u) S_1(n-i,j) S_1(i+1,m) \right) \frac{t^{n+1}}{(n+1)!}.$$

By comparing coefficients $\frac{t^n}{n!}$ on both sides of the above, we arrive at the desired result.
□

The following theorem involves in the triple summations for the products of Frobenius-Euler polynomials and Stirling numbers of the first kind.

Theorem 11. *Let $Y \sim E(\alpha)$, we have*

$$\underline{G}_n^{(k,Y)}(u) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1-i} \sum_{m=1}^{i+1} \binom{n}{i+1} \frac{\alpha^{n-i}(-1)^{n-1-i+j}}{m^{k-1}} H_j(u) S_1(n-1-i,j) S_1(i+1,m). \tag{8}$$

Proof. We omit the proof because it is same as the proof of Theorem 10. □

Remark 4. Theorem 10 is obtained as a special case of Theorem 11 in the case when $\alpha = 1$.

4. Conclusion

In the paper, we have explored a new generating function for probabilistic extensions for Frobenius-Genocchi polynomials derived from polyexponential function as follows:

$$\sum_{n=0}^{\infty} \underline{G}_n^{(k,Y)}(x,u)\frac{t^n}{n!} = \frac{(1-u)\mathrm{Ei}_k(\log(1+t))}{\mathrm{E}[e^{tY}]-u}(\mathrm{E}[e^{tY}])^x.$$

By making use of this generating function, we have also derived explicit formulae, identities and relations. Finally, by picking suitable random variables, we have establish connections between the aforementioned polynomial and other special functions and polynomials.

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References

[1] D.S. Kim and T. Kim, *A Note on Polyexponential and Unipoly Functions*. Russ. J. Math. Phys., **26**(1) (2019), 40–49. [CrossRef] [Scopus] [Web of Science]

[2] S. Aracı, *A new class of Bernoulli polynomials attached to polyexponential functions and related identities*, Adv. Stud. Contemp. Math., 2021, 195-204, **31**(2), 19 - 204. [Web]

[3] W.A. Khan, K. Ayesha and I.A. Khan, *Degenerate polyexponential-Genocchi numbers and polynomials*, J. Math. Comput. Sci, **22**(4) (2021), 381-381. [CrossRef] [Scopus] [Web of Scine]

[4] R.B. Corcino, C.B. Corcino and W.A. Khan, *Exploring Type 2 Frobenius-Type Poly-Euler Polynomials*, J. Inequal. Spec. Funct., **15**(4) (2024), 1-14. [CrossRef] [Scopus] [Web of Scine]

[5] U. Duran, M. Açıkgöz and S. Aracı, *Construction of the type 2 poly-Frobenius–Genocchi polynomials with their certain applications*, Adv. Differ. Equ., **2020**(1) (2020), 432. [CrossRef] [Scopus] [Web of Scine]

[6] T. Kim, D.S. Kim, H.Y. Kim and L.C. Jang, *Degenerate poly-Bernoulli numbers and polynomials*, Informatica, **31**(3) (2020), 2-8. [Web]

[7] P. Xue, Y. Ma, T. Kim, D.S. Kim and W. Zhang, *Probabilistic degenerate poly-Bell polynomials associated with random variables*, Math. Comput. Model. Dyn. Syst., **31**(1) (2025), 2497367. [CrossRef] [Scopus] [Web of Scine]

[8] S.H. Lee, *Probabilistic Degenerate Poly r-Stirling Numbers of the Second Kind and r-Bell Polynomials*, Eur. J. Pure Appl. Math, **18**(2) (2025), [CrossRef] [Scopus] [Web of Scine]

[9] S. Aracı, *Degenerate poly-type 2-Bernoulli polynomials*, Math. Sci. Appl. E-Notes, **9**(1) (2021), 1-8. [CrossRef] [Scopus]

[10] S. Aracı and M. Açıkgöz, *A note on the Frobenius-Euler numbers and polynomials associated with Bernstein polynomials*, Adv. Stud. Contemp. Math., **22**(3) (2012), 399-406. [CrossRef] [Scopus] [Web of Scine]

[11] M. Ma and D. Lim, *A note on polyexponential and unipoly Bernoulli polynomials of the second kind*, Open Math., **19**(1) (2021), 869-877. [CrossRef] [Scopus] [Web of Scine]

[12] J.A. Adell and A. Lekuona, *A probabilistic generalization of the Stirling numbers of the second kind*, J. Number Theory, **194**(1) (2019), 335-355. [CrossRef] [Scopus] [Web of Scine]

[13] J.A. Adell, *Probabilistic Stirling numbers of the second kind and applications*, J. Theor. Probab., **35**(1) (2022), 636-652. [CrossRef] [Scopus] [Web of Scine]

[14] T. Kim and D.S. Kim, *Explicit Formulas for Probabilistic Multi-Poly-Bernoulli Polynomials and Numbers*, Russ. J. Math. Phys., 2024, **31**(3) (2024), 450–460. [CrossRef] [Scopus] [Web of Scine]

[15] A. Karagenç, M. Açıkgöz and S. Araci *Exploring probabilistic Bernstein polynomials: identities and applications*, Appl. Math. Sci. Eng., **32**(1) (2024), 2398591. [CrossRef] [Scopus] [Web of Scine]

[16] S. Aracı and M. Açıkgöz, *Construction of Fourier expansion of Apostol Frobenius–Euler polynomials and its applications*, Adv. Differ. Equ., **2018**(1) (2018), 67. [CrossRef] [Scopus] [Web of Scine]

[17] A. Karagenç, M. Açıkgöz and S. Aracı, *A new family of q-Bernstein polynomials: Probabilistic viewpoint*, Arab J. Basic Appl. Sci., **32**(1) (2025), 42-50. [CrossRef] [Scopus]

[18] A. Karagenç, M. Açıkgöz and S. Aracı, *Poly-Frobenius-Genocchi Polynomials: A Probabilistic Perspective with Applications* Appl. Math. Inf. Sci., **19**(4) (2025), 891-898. [CrossRef] [Scopus]

[19] S.K. Sharma, W.A. Khan, S. Aracı and S.S. Ahmed, *New type of degenerate Daehee polynomials of the second kind*, Adv. Differ. Equ., **2020**(1), 428. [CrossRef] [Scopus] [Web of Scine]

[20] W.A. Khan, U. Duran and N. Ahmad, *Probabilistic degenerate Jindalrae and Jindalrae-Stirling polynomials of the second kind*, J. Nonlinear Convex Anal., **26**(4) (2025), 941-961. [Scopus] [Web of Scine]

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