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Research Paper

New Version of Simpson Type Inequality for Ψ-Hilfer Fractional Integrals

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Abstract

The ψ-Hilfer fractional integrals serve as a generalization encompassing well-known fractional integrals like Riemann-Liouville and Hadamard fractional integrals. This investigation initiates by demonstrating a key identity tied to ψ-Hilfer fractional integrals, specifically tailored for differentiable functions. Leveraging this identity, we establish a series of Simpson-type inequalities applicable to ψ-Hilfer fractional integrals. To achieve this, we delve into the realms of convexity and the renowned Hölder inequality. Furthermore, we explore the correlations between our primary discoveries and preceding research endeavors.

Key Words: Simpson's Type Inequality, Integral Inequalities, Bounded Functions
AMS 2020 Classification: 26D07, 26D10, 26D15, 26A33

1. Introduction

It is well known that the considerable number of inequalities have been established in the case of convex functions but the most famous is Simpson's inequality. The classical Simpson's inequality for four times continuously differentiable functions are expressed as follows:

Theorem 1. *Let us note that $\mathcal{F} : [\sigma, \rho] \to \mathbb{R}$ is a four times continuously differentiable function on (σ, ρ), and let us consider $\left\|\mathcal{F}^{(4)}\right\|_\infty = \sup_{\kappa \in (\sigma,\rho)} \left|\mathcal{F}^{(4)}(\kappa)\right| < \infty$. Then, the following inequality holds:*

$$\left|\frac{1}{6}\left[\mathcal{F}(\sigma) + 4\mathcal{F}\left(\frac{\sigma+\rho}{2}\right) + \mathcal{F}(\rho)\right] - \frac{1}{\rho-\sigma}\int_\sigma^\rho \mathcal{F}(\kappa)d\kappa\right| \le \frac{1}{2880}\left\|\mathcal{F}^{(4)}\right\|_\infty (\rho-\sigma)^4 .$$

Since the convex theory is an effective and useful way to solve a large number of problems from different branches of mathematics, many mathematicians have investigated the Simpson-type inequalities the case of convex function. More precisely, some inequalities of Simpson's type for s-convex functions are established by using differentiable functions in the paper [1]. Furthermore, the new variants of Simpson's type inequalities based on differentiable convex functions are established in the papers [2]. The reader is referred to [3, 4, 5] and the references therein for more information and unexplained subjects about Simpson type inequalities for various convex classes.

Many mathematicians have investigated the twice differentiable convex functions for obtaining significant inequalities. For example, Sarikaya et al. proved several Simpson-type inequalities for functions whose second derivatives are convex in the paper [6]. In addition, some Simpson's type inequalities are given for functions whose absolute values of derivatives are convex in the paper [7]. Furthermore, Simpson type inequalities are established for P-convex functions in the paper [8]. It can be referred to [9, 10, 11, 12] for further pieces of informations and unexplained subjects about these type of inequalities including twice differentiable functions.

Mathematical preliminaries about fractional calculus theory, which will be used throughout this paper, will be given as follows:

Definition 1. Let us consider $\mathcal{F} \in L_1[\sigma, \rho]$. The Riemann–Liouville integrals $J^{\alpha}_{\sigma+}\mathcal{F}$ and $J^{\alpha}_{\rho-}\mathcal{F}$ of order $\alpha > 0$ with $\sigma \geq 0$ are defined by

$$J^{\alpha}_{\sigma+}\mathcal{F}(\kappa) = \frac{1}{\Gamma(\alpha)} \int_{\sigma}^{\kappa} (\kappa - \zeta)^{\alpha-1} \mathcal{F}(\zeta) d\zeta, \quad \kappa > \sigma \tag{1}$$

and

$$J^{\alpha}_{\rho-}\mathcal{F}(\kappa) = \frac{1}{\Gamma(\alpha)} \int_{\kappa}^{\rho} (\zeta - \kappa)^{\alpha-1} \mathcal{F}(\zeta) d\zeta, \quad \kappa < \rho, \tag{2}$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and its described as follows:

$$\Gamma(\alpha) = \int_{0}^{\infty} e^{-u} u^{\alpha-1} du.$$

Let us also note that $J^{0}_{\sigma+}\mathcal{F}(\kappa) = J^{0}_{\rho-}\mathcal{F}(\kappa) = \mathcal{F}(\kappa)$.

Remark 1. If we choose $\alpha = 1$ in Definition 1, then the fractional integral reduces to the classical integral.

Definition 2. Let $\mathcal{F} \in L_1[\sigma, \rho]$. The Hadamard fractional integrals $\mathbf{J}^{\alpha}_{\sigma+}\mathcal{F}$ and $\mathbf{J}^{\alpha}_{\rho-}\mathcal{F}$ of order $\alpha > 0$ with $\sigma \geq 0$ are defined by

$$\mathbf{J}^{\alpha}_{\sigma+}\mathcal{F}(\kappa) = \frac{1}{\Gamma(\alpha)} \int_{\sigma}^{\kappa} \left(\ln \frac{\kappa}{\zeta} \right)^{\alpha-1} \mathcal{F}(\zeta) \frac{d\zeta}{\zeta}, \quad \kappa > \sigma \tag{3}$$

and

$$\mathbf{J}^{\alpha}_{\rho-}\mathcal{F}(\kappa) = \frac{1}{\Gamma(\alpha)} \int_{\kappa}^{\rho} \left(\ln \frac{\zeta}{\kappa} \right)^{\alpha-1} \mathcal{F}(\zeta) \frac{d\zeta}{\zeta}, \quad \kappa < \rho \tag{4}$$

respectively.

In the paper [13], the Simpson inequalities for differentiable functions are extended to Riemann-Liouville fractional integrals. In addition to these, several papers are focused on fractional Simpson inequalities for fractional and various fractional integral operators [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24]. For further information and several properties of Riemann–Liouville fractional integrals, please refer to [25, 26, 27, 28].

Theorem 2. *[29] Let consider that $\mathcal{F} : [\sigma, \rho] \to \mathbb{R}$ is a differentiable on (σ, ρ). If $|\mathcal{F}'|$ is a convex function on $[\sigma, \rho]$, then the following inequalities hold:*

$$\begin{aligned} & \left| \frac{1}{6} \left[\mathcal{F}(\sigma) + 4\mathcal{F}\left(\frac{\sigma+\rho}{2} \right) + \mathcal{F}(\rho) \right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\rho-\sigma)^{\alpha}} \left[J^{\alpha}_{\frac{\sigma+\rho}{2}+}\mathcal{F}(\rho) + J^{\alpha}_{\frac{\sigma+\rho}{2}-}\mathcal{F}(\sigma) \right] \right| \\ \leq\ & \frac{\rho-\sigma}{12} \left[C_2(\alpha) \left(\left|\mathcal{F}'(\rho)\right| + \left|\mathcal{F}'(\sigma)\right| \right) + 2C_1(\alpha) \left| \mathcal{F}'\left(\frac{\sigma+\rho}{2} \right) \right| \right] \end{aligned}$$

$$\leq \frac{\rho-\sigma}{12}\left(C_1(\alpha)+C_2(\alpha)\right)\left(\left|\mathcal{F}'(\sigma)\right|+\left|\mathcal{F}'(\rho)\right|\right),$$

where

$$C_1(\alpha) = 2\left(\frac{1}{3}\right)^{\frac{2}{\alpha}}\left[\frac{1}{2}-\frac{1}{\alpha+2}\right]+\frac{3}{\alpha+2}-\frac{1}{2},$$

$$C_2(\alpha) = 2\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}\left[1-\frac{1}{\alpha+1}\right]-2\left(\frac{1}{3}\right)^{\frac{2}{\alpha}}\left[\frac{1}{2}-\frac{1}{\alpha+2}\right]+\frac{3}{(\alpha+1)(\alpha+2)}-\frac{1}{2}.$$

The definitions of the following ψ-Hilfer fractional integrals are given in [25].

Definition 3. Let $\psi:[\sigma,\rho]\to\mathbb{R}$ be an monotone increasing function on $(\sigma,\rho]$, having a continuous derivative $\psi'(\kappa)$ on (σ,ρ). The left-sided and right sided fractional integrals of $\mathcal{F}$ with respect to the function ψ on $[\sigma,\rho]$ of order $\alpha>0$ are defined by

$$I^{\alpha}_{\sigma+;\psi}\mathcal{F}(\kappa)=\frac{1}{\Gamma(\alpha)}\int_{\sigma}^{\kappa}\left(\psi(\kappa)-\psi(\zeta)\right)^{\alpha-1}\psi'(\zeta)\mathcal{F}(\zeta)d\zeta,\kappa>\sigma \tag{5}$$

and

$$I^{\alpha}_{\rho-;\psi}\mathcal{F}(\kappa)=\frac{1}{\Gamma(\alpha)}\int_{\kappa}^{\rho}\left(\psi(\zeta)-\psi(\kappa)\right)^{\alpha-1}\psi'(\zeta)\mathcal{F}(\zeta)d\zeta,\kappa<\rho \tag{6}$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function.

Remark 2. If we choose $\psi(\zeta)=\zeta$, then the operators (5) and (6) reduce the Riemann-Liouville fractional operators (1) and (2), respectively.

Remark 3. If we choose $\psi(\zeta)=\ln\zeta$, $\zeta>0$, then the operators (5) and (6) reduce the Hadamard fractional operators (3) and (4), respectively.

2. Simpson Type Inequalities for Ψ-Hilfer Fractional Integrals

In this section we will prove some new versions of Simpson type inequalities for Ψ-Hilfer fractional integrals.

For the sake of brevity, we denote

$$M^{\alpha}_{\psi}(\sigma,\rho)=\left[\psi\left(\frac{\sigma+\rho}{2}\right)-\psi(\sigma)\right]^{\alpha}$$

and

$$N^{\alpha}_{\psi}(\sigma,\rho)=\left[\psi(\rho)-\psi\left(\frac{\sigma+\rho}{2}\right)\right]^{\alpha}.$$

In particularly, let $\psi(\zeta)=\ln\zeta$, $\zeta>0$. Then we have

$$M^{\alpha}_{\ln}(\sigma,\rho)=\left[\ln\left(\frac{\sigma+\rho}{2\sigma}\right)\right]^{\alpha}$$

and

$$N^{\alpha}_{\ln}(\sigma,\rho)=\left[\ln\left(\frac{2\rho}{\sigma+\rho}\right)\right]^{\alpha}.$$

Lemma 1. Assume that $\mathcal{F}:[\sigma,\rho]\to\mathbb{R}$ is a differentiable on (σ,ρ). The we have the following identity

$$\frac{1}{6}\left[\mathcal{F}(\sigma)+4\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+\mathcal{F}(\rho)\right]-\frac{\Gamma(\alpha+1)}{2}\left[\frac{1}{M^{\alpha}_{\psi}(\sigma,\rho)}I^{\alpha}_{\frac{\sigma+\rho}{2}-;\psi}\mathcal{F}(\sigma)+\frac{1}{N^{\alpha}_{\psi}(\sigma,\rho)}I^{\alpha}_{\frac{\sigma+\rho}{2}+;\psi}\mathcal{F}(\rho)\right] \tag{7}$$

$$= \frac{\rho-\sigma}{4}\left[\frac{1}{M_{\psi}^{\alpha}(\sigma,\rho)}\int_{0}^{1}\Delta_{\psi,\alpha}(\zeta)\mathcal{F}'\left(\frac{2-\zeta}{2}\sigma+\frac{\zeta}{2}\rho\right)d\zeta-\frac{1}{N_{\psi}^{\alpha}(\sigma,\rho)}\int_{0}^{1}\Lambda_{\psi,\alpha}(\zeta)\mathcal{F}'\left(\frac{2-\zeta}{2}\rho+\frac{\zeta}{2}\sigma\right)d\zeta\right],$$

where

$$\Lambda_{\psi,\alpha}(\zeta)=\left[\psi(\rho)-\psi\left(\frac{2-\zeta}{2}\rho+\frac{\zeta}{2}\sigma\right)\right]^{\alpha}-\frac{1}{3}N_{\psi}^{\alpha}(\sigma,\rho)$$

and

$$\Delta_{\psi,\alpha}(\zeta)=\left[\psi\left(\frac{2-\zeta}{2}\sigma+\frac{\zeta}{2}\rho\right)-\psi(\sigma)\right]^{\alpha}-\frac{1}{3}M_{\psi}^{\alpha}(\sigma,\rho).$$

Proof. By using integration by parts, we have

$$\begin{aligned}\int_{0}^{1}\Lambda_{\psi,\alpha}(\zeta)\mathcal{F}'\left(\frac{2-\zeta}{2}\rho+\frac{\zeta}{2}\sigma\right)d\zeta &= -\frac{2}{\rho-\sigma}\Lambda_{\psi,\alpha}(\zeta)\mathcal{F}\left(\frac{2-\zeta}{2}\rho+\frac{\zeta}{2}\sigma\right)\Big|_{0}^{1}+\alpha\int_{0}^{1}\left[\psi(\rho)-\psi\left(\frac{2-\zeta}{2}\rho+\frac{\zeta}{2}\sigma\right)\right]^{\alpha-1}\\ &\quad\times\psi'\left(\frac{2-\zeta}{2}\rho+\frac{\zeta}{2}\sigma\right)\mathcal{F}\left(\frac{2-\zeta}{2}\rho+\frac{\zeta}{2}\sigma\right)d\zeta \qquad (8)\\ &= -\frac{2}{\rho-\sigma}\left[\frac{2}{3}N_{\psi}^{\alpha}(\sigma,\rho)\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+\frac{1}{3}N_{\psi}^{\alpha}(\sigma,\rho)\mathcal{F}(\rho)\right]\\ &\quad+\frac{2\alpha}{\rho-\sigma}\int_{\frac{\sigma+\rho}{2}}^{\rho}\left[\psi(\rho)-\psi(\kappa)\right]^{\alpha-1}\psi'(\kappa)\mathcal{F}(\kappa)\,d\kappa\\ &= \frac{2N_{\psi}^{\alpha}(\sigma,\rho)}{\rho-\sigma}\left[\frac{1}{3}\mathcal{F}(\rho)+\frac{2}{3}\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)\right]+\frac{2\Gamma(\alpha+1)}{\rho-\sigma}I_{\frac{\sigma+\rho}{2}+;\psi}^{\alpha}\mathcal{F}(\rho).\end{aligned}$$

Similarly, we get

$$\int_{0}^{1}\Delta_{\psi,\alpha}(\zeta)\mathcal{F}'\left(\frac{2-\zeta}{2}\sigma+\frac{\zeta}{2}\rho\right)d\zeta=\frac{2M_{\psi}^{\alpha}(\sigma,\rho)}{\rho-\sigma}\left[\frac{1}{3}\mathcal{F}(\sigma)+\frac{2}{3}\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)\right]-\frac{2\Gamma(\alpha+1)}{\rho-\sigma}I_{\frac{\sigma+\rho}{2}-;\psi}^{\alpha}\mathcal{F}(\sigma). \qquad (9)$$

If we multiply both sides of (8) and (9) by $\frac{-(\rho-\sigma)}{4N_{\psi}^{\alpha}(\sigma,\rho)}$ and $\frac{\rho-\sigma}{4N_{\psi}^{\alpha}(\sigma,\rho)}$, respectively, then by adding the resultant equalities, the equality (7) is obtained. This finishes the proof of Lemma 1. □

Theorem 3. *Let consider that the assumptions of Lemma 1 hold. If the function $|\mathcal{F}'|$ is convex on $[\sigma,\rho]$, then one has the inequality*

$$\begin{aligned}&\left|\frac{1}{6}\left[\mathcal{F}(\sigma)+4\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+\mathcal{F}(\rho)\right]-\frac{\Gamma(\alpha+1)}{2}\left[\frac{1}{M_{\psi}^{\alpha}(\sigma,\rho)}I_{\frac{\sigma+\rho}{2}-;\psi}^{\alpha}\mathcal{F}(\sigma)+\frac{1}{N_{\psi}^{\alpha}(\sigma,\rho)}I_{\frac{\sigma+\rho}{2}+;\psi}^{\alpha}\mathcal{F}(\rho)\right]\right|\\ &\leq \frac{\rho-\sigma}{8}\left[\frac{2\Omega_{1}(\psi,\alpha)-\Omega_{2}(\psi,\alpha)}{M_{\psi}^{\alpha}(\sigma,\rho)}\left|\mathcal{F}'(\sigma)\right|+\frac{\Omega_{2}(\psi,\alpha)}{M_{\psi}^{\alpha}(\sigma,\rho)}\left|\mathcal{F}'(\rho)\right|\right.\end{aligned}$$

$$+\frac{2\Upsilon_1(\psi,\alpha)-\Upsilon_2(\psi,\alpha)}{N_\psi^\alpha(\sigma,\rho)}\left|\mathcal{F}'(\rho)\right|+\frac{\Upsilon_2(\psi,\alpha)}{N_\psi^\alpha(\sigma,\rho)}\left|\mathcal{F}'(\sigma)\right|\Bigg],$$

where

$$\Omega_1(\psi,\alpha)=\int_0^1\left|\Delta_{\psi,\alpha}(\zeta)\right|d\zeta,$$

$$\Omega_2(\psi,\alpha)=\int_0^1\zeta\left|\Delta_{\psi,\alpha}(\zeta)\right|d\zeta,$$

$$\Upsilon_1(\psi,\alpha)=\int_0^1\left|\Lambda_{\psi,\alpha}(\zeta)\right|d\zeta$$

and

$$\Upsilon_2(\psi,\alpha)=\int_0^1\zeta\left|\Lambda_{\psi,\alpha}(\zeta)\right|d\zeta.$$

Proof. Let us take modulus in Lemma 1. Then, we have

$$\left|\frac{1}{6}\left[\mathcal{F}(\sigma)+4\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+\mathcal{F}(\rho)\right]-\frac{\Gamma(\alpha+1)}{2}\left[\frac{1}{M_\psi^\alpha(\sigma,\rho)}I^\alpha_{\frac{\sigma+\rho}{2}-;\psi}\mathcal{F}(\sigma)+\frac{1}{N_\psi^\alpha(\sigma,\rho)}I^\alpha_{\frac{\sigma+\rho}{2}+;\psi}\mathcal{F}(\rho)\right]\right| \tag{10}$$

$$\leq \frac{\rho-\sigma}{4}\left[\frac{1}{M_\psi^\alpha(\sigma,\rho)}\int_0^1\left|\Delta_{\psi,\alpha}(\zeta)\right|\left|\mathcal{F}'\left(\frac{2-\zeta}{2}\sigma+\frac{\zeta}{2}\rho\right)\right|d\zeta+\frac{1}{N_\psi^\alpha(\sigma,\rho)}\int_0^1\left|\Lambda_{\psi,\alpha}(\zeta)\right|\left|\mathcal{F}'\left(\frac{2-\zeta}{2}\rho+\frac{\zeta}{2}\sigma\right)\right|d\zeta\right].$$

Using the fact that $\left|\mathcal{F}'\right|$ is convex, we obtain

$$\left|\frac{1}{6}\left[\mathcal{F}(\sigma)+4\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+\mathcal{F}(\rho)\right]-\frac{\Gamma(\alpha+1)}{2}\left[\frac{1}{M_\psi^\alpha(\sigma,\rho)}I^\alpha_{\frac{\sigma+\rho}{2}-;\psi}\mathcal{F}(\sigma)+\frac{1}{N_\psi^\alpha(\sigma,\rho)}I^\alpha_{\frac{\sigma+\rho}{2}+;\psi}\mathcal{F}(\rho)\right]\right|$$

$$\leq \frac{\rho-\sigma}{4}\left[\frac{1}{M_\psi^\alpha(\sigma,\rho)}\int_0^1\left|\Delta_{\psi,\alpha}(\zeta)\right|\left[\frac{2-\zeta}{2}\left|\mathcal{F}'(\sigma)\right|+\frac{\zeta}{2}\left|\mathcal{F}'(\rho)\right|\right]d\zeta\right.$$

$$\left.+\frac{1}{N_\psi^\alpha(\sigma,\rho)}\int_0^1\left|\Lambda_{\psi,\alpha}(\zeta)\right|\left[\frac{2-\zeta}{2}\left|\mathcal{F}'(\rho)\right|+\frac{\zeta}{2}\left|\mathcal{F}'(\sigma)\right|\right]d\zeta\right]$$

$$=\frac{\rho-\sigma}{8}\left[\frac{2\Omega_1(\psi,\alpha)-\Omega_2(\psi,\alpha)}{M_\psi^\alpha(\sigma,\rho)}\left|\mathcal{F}'(\sigma)\right|+\frac{\Omega_2(\psi,\alpha)}{M_\psi^\alpha(\sigma,\rho)}\left|\mathcal{F}'(\rho)\right|\right.$$

$$\left.+\frac{2\Upsilon_1(\psi,\alpha)-\Upsilon_2(\psi,\alpha)}{N_\psi^\alpha(\sigma,\rho)}\left|\mathcal{F}'(\rho)\right|+\frac{\Upsilon_2(\psi,\alpha)}{N_\psi^\alpha(\sigma,\rho)}\left|\mathcal{F}'(\sigma)\right|\right].$$

This finishes the proof of Theorem 3. □

Remark 4. Let us consider $\psi(\zeta)=\zeta$ for all $\zeta\in[\sigma,\rho]$ in Theorem 3. Then, we obtain

$$\Omega_1(\psi,\alpha) = \Upsilon_1(\psi,\alpha)=\left(\frac{\rho-\sigma}{2}\right)^{\alpha}\int_0^1\left|\zeta^{\alpha}-\frac{1}{3}\right|d\zeta=\left(\frac{\rho-\sigma}{2}\right)^{\alpha}\left[\frac{2\alpha}{\alpha+1}\left(\frac{1}{3}\right)^{1+\frac{1}{\alpha}}+\frac{2-\alpha}{3(\alpha+1)}\right],$$

$$\Omega_2(\psi,\alpha) = \Upsilon_2(\psi,\alpha)=\left(\frac{\rho-\sigma}{2}\right)^{\alpha}\int_0^1\zeta\left|\zeta^{\alpha}-\frac{1}{3}\right|d\zeta=\left(\frac{\rho-\sigma}{2}\right)^{\alpha}\left[\frac{\alpha}{\alpha+2}\left(\frac{1}{3}\right)^{1+\frac{2}{\alpha}}+\frac{4-\alpha}{6(\alpha+2)}\right],$$

and

$$\left|\frac{1}{6}\left[\mathcal{F}(\sigma)+4\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+\mathcal{F}(\rho)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\rho-\sigma)^{\alpha}}\left[J^{\alpha}_{\frac{\sigma+\rho}{2}-}\mathcal{F}(\sigma)+J^{\alpha}_{\frac{\sigma+\rho}{2}+}\mathcal{F}(\rho)\right]\right|$$

$$\leq \frac{\rho-\sigma}{4}\left[\frac{2\alpha}{\alpha+1}\left(\frac{1}{3}\right)^{1+\frac{1}{\alpha}}+\frac{2-\alpha}{3(\alpha+1)}\right]\left[\left|\mathcal{F}'(\sigma)\right|+\left|\mathcal{F}'(\rho)\right|\right].$$

This result equivalent to Theorem 2.

Corollary 1. Let us consider $\psi(\zeta)=\ln\zeta$, $\zeta>0$ in Theorem 3. Then, we obtain the following Simpson type inequalities for Hadamard fractiona lintegrals

$$\left|\frac{1}{6}\left[\mathcal{F}(\sigma)+4\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+\mathcal{F}(\rho)\right]-\frac{\Gamma(\alpha+1)}{2}\left[\frac{1}{M^{\alpha}_{\ln}(\sigma,\rho)}\mathbf{J}^{\alpha}_{\frac{\sigma+\rho}{2}-}\mathcal{F}(\sigma)+\frac{1}{N^{\alpha}_{\ln}(\sigma,\rho)}\mathbf{J}^{\alpha}_{\frac{\sigma+\rho}{2}+}\mathcal{F}(\rho)\right]\right|$$

$$\leq \frac{\rho-\sigma}{8}\left[\frac{2\Omega_3(\psi,\alpha)-\Omega_4(\psi,\alpha)}{M^{\alpha}_{\ln}(\sigma,\rho)}\left|\mathcal{F}'(\sigma)\right|+\frac{\Omega_4(\psi,\alpha)}{M^{\alpha}_{\ln}(\sigma,\rho)}\left|\mathcal{F}'(\rho)\right|\right.$$

$$\left.+\frac{2\Upsilon_3(\psi,\alpha)-\Upsilon_4(\psi,\alpha)}{N^{\alpha}_{\psi}(\sigma,\rho)}\left|\mathcal{F}'(\rho)\right|+\frac{\Upsilon_4(\psi,\alpha)}{N^{\alpha}_{\psi}(\sigma,\rho)}\left|\mathcal{F}'(\sigma)\right|\right],$$

where

$$\Omega_3(\ln,\alpha)=\int_0^1\left|\Delta_{\ln,\alpha}(\zeta)\right|d\zeta,$$

$$\Omega_4(\ln,\alpha)=\int_0^1\zeta\left|\Delta_{\ln,\alpha}(\zeta)\right|d\zeta,$$

$$\Upsilon_3(\ln,\alpha)=\int_0^1\left|\Lambda_{\ln,\alpha}(\zeta)\right|d\zeta$$

and

$$\Upsilon_4(\ln,\alpha)=\int_0^1\zeta\left|\Lambda_{\ln,\alpha}(\zeta)\right|d\zeta$$

Theorem 4. *Let consider that the assumptions of Lemma 1 hold. If the function* $\left|\mathcal{F}'\right|^{q}$, $q>1$, *is convex on* $[\sigma,\rho]$, *then one has the following Simpson inequality*

$$\left|\frac{1}{6}\left[\mathcal{F}(\sigma)+4\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+\mathcal{F}(\rho)\right]-\frac{\Gamma(\alpha+1)}{2}\left[\frac{1}{M^{\alpha}_{\psi}(\sigma,\rho)}I^{\alpha}_{\frac{\sigma+\rho}{2}-;\psi}\mathcal{F}(\sigma)+\frac{1}{N^{\alpha}_{\psi}(\sigma,\rho)}I^{\alpha}_{\frac{\sigma+\rho}{2}+;\psi}\mathcal{F}(\rho)\right]\right|$$

$$\leq \frac{\rho-\sigma}{4M_{\psi}^{\alpha}(\sigma,\rho)}\left(\int_{0}^{1}|\Delta_{\psi,\alpha}(\zeta)|^{p}\,d\zeta\right)^{\frac{1}{p}}\left(\frac{3\left|\mathcal{F}'(\sigma)\right|^{q}+\left|\mathcal{F}'(\rho)\right|^{q}}{4}\right)^{\frac{1}{q}} \tag{11}$$

$$+\frac{\rho-\sigma}{4N_{\psi}^{\alpha}(\sigma,\rho)}\left(\int_{0}^{1}|\Lambda_{\psi,\alpha}(\zeta)|^{p}\,d\zeta\right)^{\frac{1}{p}}\left(\frac{\left|\mathcal{F}'(\sigma)\right|^{q}+3\left|\mathcal{F}'(\rho)\right|^{q}}{4}\right)^{\frac{1}{q}}$$

$$\leq \left[\frac{\rho-\sigma}{4M_{\psi}^{\alpha}(\sigma,\rho)}\left(4\int_{0}^{1}|\Delta_{\psi,\alpha}(\zeta)|^{p}\,d\zeta\right)^{\frac{1}{p}}+\frac{\rho-\sigma}{4N_{\psi}^{\alpha}(\sigma,\rho)}\left(4\int_{0}^{1}|\Lambda_{\psi,\alpha}(\zeta)|^{p}\,d\zeta\right)^{\frac{1}{p}}\right]\left[\left|\mathcal{F}'(\sigma)\right|+\left|\mathcal{F}'(\rho)\right|\right],$$

where $\frac{1}{p}+\frac{1}{q}=1.$

Proof. By utilizing the well-known Hölder inequality and convexity of $\left|\mathcal{F}'\right|^{q}$, we obtain

$$\begin{aligned}\int_{0}^{1}|\Delta_{\psi,\alpha}(\zeta)|\left|\mathcal{F}'\left(\frac{2-\zeta}{2}\sigma+\frac{\zeta}{2}\rho\right)\right|d\zeta &\leq \left(\int_{0}^{1}|\Delta_{\psi,\alpha}(\zeta)|^{p}\,d\zeta\right)^{\frac{1}{p}}\left(\int_{0}^{1}\left|\mathcal{F}'\left(\frac{2-\zeta}{2}\sigma+\frac{\zeta}{2}\rho\right)\right|^{q}d\zeta\right)^{\frac{1}{q}} \qquad (12)\\ &\leq \left(\int_{0}^{1}|\Delta_{\psi,\alpha}(\zeta)|^{p}\,d\zeta\right)^{\frac{1}{p}}\left(\int_{0}^{1}\left[\frac{2-\zeta}{2}\left|\mathcal{F}'(\sigma)\right|^{q}+\frac{\zeta}{2}\left|\mathcal{F}'(\rho)\right|^{q}\right]d\zeta\right)^{\frac{1}{q}}\\ &= \left(\int_{0}^{1}|\Delta_{\psi,\alpha}(\zeta)|^{p}\,d\zeta\right)^{\frac{1}{p}}\left(\frac{3\left|\mathcal{F}'(\sigma)\right|^{q}+\left|\mathcal{F}'(\rho)\right|^{q}}{4}\right)^{\frac{1}{q}},\end{aligned}$$

and similarly

$$\int_{0}^{1}|\Lambda_{\psi,\alpha}(\zeta)|\left|\mathcal{F}'\left(\frac{2-\zeta}{2}\rho+\frac{\zeta}{2}\sigma\right)\right|d\zeta\leq\left(\int_{0}^{1}|\Lambda_{\psi,\alpha}(\zeta)|^{p}\,d\zeta\right)^{\frac{1}{p}}\left(\frac{\left|\mathcal{F}'(\sigma)\right|^{q}+3\left|\mathcal{F}'(\rho)\right|^{q}}{4}\right)^{\frac{1}{q}}. \tag{13}$$

By (12) and (13) in (10), we establish

$$\left|\frac{1}{6}\left[\mathcal{F}(\sigma)+4\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+\mathcal{F}(\rho)\right]-\frac{\Gamma(\alpha+1)}{2}\left[\frac{1}{M_{\psi}^{\alpha}(\sigma,\rho)}I^{\alpha}_{\frac{\sigma+\rho}{2}-;\psi}\mathcal{F}(\sigma)+\frac{1}{N_{\psi}^{\alpha}(\sigma,\rho)}I^{\alpha}_{\frac{\sigma+\rho}{2}+;\psi}\mathcal{F}(\rho)\right]\right|$$

$$\leq \frac{\rho-\sigma}{4}\left[\frac{1}{M_{\psi}^{\alpha}(\sigma,\rho)}\left(\int_{0}^{1}|\Delta_{\psi,\alpha}(\zeta)|^{p}\,d\zeta\right)^{\frac{1}{p}}\left(\frac{3\left|\mathcal{F}'(\sigma)\right|^{q}+\left|\mathcal{F}'(\rho)\right|^{q}}{4}\right)^{\frac{1}{q}}\right.$$

$$+\frac{1}{N_{\psi}^{\alpha}(\sigma,\rho)}\left(\int_{0}^{1}\left|\Lambda_{\psi,\alpha}(\zeta)\right|^{p}d\zeta\right)^{\frac{1}{p}}\left(\frac{\left|\mathcal{F}'(\sigma)\right|^{q}+3\left|\mathcal{F}'(\rho)\right|^{q}}{4}\right)^{\frac{1}{q}}\Bigg].$$

This gives the first inequality in (11). For the proof of second inequality, let $a_1 = 3\left|\mathcal{F}'(a)\right|^q$, $b_1 = \left|\mathcal{F}'(b)\right|^q$, $a_2 = \left|\mathcal{F}'(a)\right|^q$ and $b_2 = 3\left|\mathcal{F}'(b)\right|^q$. Using the facts that,

$$\sum_{k=1}^{n}(a_k+b_k)^s \leq \sum_{k=1}^{n}a_k^s + \sum_{k=1}^{n}b_k^s,\ 0\leq s<1 \tag{14}$$

and $1+3^{\frac{1}{q}} \leq 4$, then the desired result can be obtained straightforwardly. This completes the proof of Theorem 4. □

Remark 5. Let us consider $\psi(\zeta)=\zeta$ for all $\zeta\in[\sigma,\rho]$ in Theorem 4. Then, we obtain

$$\left|\frac{1}{6}\left[\mathcal{F}(\sigma)+4\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+\mathcal{F}(\rho)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\rho-\sigma)^{\alpha}}\left[J_{\frac{\sigma+\rho}{2}-}^{\alpha}\mathcal{F}(\sigma)+J_{\frac{\sigma+\rho}{2}+}^{\alpha}\mathcal{F}(\rho)\right]\right|$$

$$\leq \frac{\rho-\sigma}{4}\left(\int_{0}^{1}\left|\zeta^{\alpha}-\frac{1}{3}\right|^{p}d\zeta\right)^{\frac{1}{p}}\left(\frac{3\left|\mathcal{F}'(\sigma)\right|^{q}+\left|\mathcal{F}'(\rho)\right|^{q}}{4}\right)^{\frac{1}{q}}$$

$$+\frac{\rho-\sigma}{4}\left(\int_{0}^{1}\left|\zeta^{\alpha}-\frac{1}{3}\right|^{p}d\zeta\right)^{\frac{1}{p}}\left(\frac{\left|\mathcal{F}'(\sigma)\right|^{q}+3\left|\mathcal{F}'(\rho)\right|^{q}}{4}\right)^{\frac{1}{q}}$$

$$\leq \frac{\rho-\sigma}{2}\left(4\int_{0}^{1}\left|\zeta^{\alpha}-\frac{1}{3}\right|^{p}d\zeta\right)^{\frac{1}{p}}\left[\left|\mathcal{F}'(\sigma)\right|+\left|\mathcal{F}'(\rho)\right|\right].$$

3. Conclusion

In this work, we first prove an identity involving ψ-Hilfer fractional integrals for differentiable functions. By utilizing this equality, we obtain some Simpson-type inequalities for ψ-Hilfer fractional integrals. For this aim, we consider the concept of convexity and well-known Hölder inequality. Moreover, we discuss the connections between our main findings and previous studies.

In conclusion, our study has shed light on the fascinating world of ψ-Hilfer fractional integrals and their associated Simpson-type inequalities. We hope that this work will inspire further exploration and research in this area, potentially leading to new applications and insights in various scientific disciplines and real-world problem-solving.

Declarations

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References

[1] M. Alomari, M. Darus, and S.S. Dragomir, *New inequalities of Simpson's type for s-convex functions with applications*, RGMIA Res. Rep. Coll., **12**(4) (2009), 1-18. [Web]

[2] M.Z. Sarıkaya, E. Set and M.E. Özdemir, *On new inequalities of Simpson's type for convex functions*, Comput. Math. Appl., **60**(8) (2010), 2191-2199. [CrossRef] [Scopus] [Web of Science]

[3] T. Du, Y. Li and Z. Yang, *A generalization of Simpson's inequality via differentiable mapping using extended (s, m)-convex functions*, Appl. Math. Comput., **293**(15) (2017), 358-369. [CrossRef] [Scopus] [Web of Science]

[4] İ. İşcan, *Hermite-Hadamard and Simpson-like type inequalities for differentiable harmonically convex functions*, J. Math., **2014** (2014), Article ID 346305, 1-10. [CrossRef] [Scopus] [Web of Science]

[5] M.Z. Sarıkaya and S. Bardak, *Generalized Simpson type integral inequalities*, Konuralp J. Math., **7**(1) (2019) 186-191. [DergiPark]

[6] M.Z. Sarıkaya, E. Set and M.E. Özdemir, *On new inequalities of Simpson's type for functions whose second derivatives absolute values are convex*, J. Appl. Math. Stat. Inform., **9**(1) (2013), 37-45. [CrossRef] [Web of Science]

[7] M.Z. Sarıkaya and N. Aktan, *On the generalization of some integral inequalities and their applications*, Math. Comput. Model., **54**(9-10) (2011), 2175-2182. [CrossRef] [Scopus] [Web of Science]

[8] M.E. Özdemir and Ç. Yıldız, *New inequalities for Hermite-Hadamard and Simpson type with applications*, Tamkang J. Math., **44**(2) (2013), 209-216. [CrossRef] [Web of Science]

[9] M.A. Ali, H. Kara, J. Tariboon, S. Asawasamrit, H. Budak and F. Hezenci, *Some new Simpson's-Formula-Type inequalities for twice–differentiable convex functions via generalized fractional operators*, Symmetry, **13**(12), (2021), 2249. [CrossRef] [Scopus] [Web of Science]

[10] F. Hezenci, H. Budak and H. Kara, *New version of Fractional Simpson type inequalities for twice differentiable functions*, Adv. Differ. Equ., **2021**(460) (2021), 1-10. [CrossRef] [Scopus] [Web of Science]

[11] X. You, F. Hezenci, H. Budak and H. Kara, *New Simpson type inequalities for twice differentiable functions via generalized fractional integrals*, AIMS Math., **7**(3) (2021), 3959-3971. [CrossRef] [Scopus] [Web of Science]

[12] J. Park, *On Simpson-like type integral inequalities for differentiable preinvex functions*, Appl. Math. Sci., **7**(121) 2013, 6009-6021. [CrossRef] [Scopus]

[13] M. Iqbal, S. Qaisar and S. Hussain, *On Simpson's type inequalities utilizing fractional integrals*, J. Comput. Anal. Appl., **23**(6) (2017), 1137-1145. [Scopus] [Web of Science]

[14] P. Agarwal, M. Vivas-Cortez, Y. Rangel-Oliveros, and M.A. Ali, *New Ostrowski type inequalities for generalized s-convex functions with applications to some special means of real numbers and to midpoint formula*, AIMS Math., **7**(1), (2022), 1429-1444. [CrossRef] [Scopus] [Web of Science]

[15] M.A. Ali, M. Abbas, H. Budak, P. Agarwal, G. Murtaza and Y.M. Chu, *New quantum boundaries for quantum Simpson's and quantum Newton's type inequalities for preinvex functions*, Adv. Differ. Equ., **2021**(64) (2021), 1-21. [CrossRef] [Scopus] [Web of Science]

[16] S.I. Butt, I. Javed, P. Agarwal and J.J. Nieto, *Newton–Simpson-type inequalities via majorization*, J. Inequal. Appl., **2023**(16) (2023), 1-34. [CrossRef] [Scopus] [Web of Science]

[17] J. Chen and X. Huang, *Some new inequalities of Simpson's type for s-convex functions via fractional integrals*, Filomat, **31**(15 (2017), 4989–4997. [CrossRef] [Scopus] [Web of Science]

[18] İ. Demir, *A new approach of Milne-type inequalities based on proportional Caputo-Hybrid operator*, J. Adv. Appl. Computat. Math., **10** (2023), 102-119. [CrossRef]

[19] H.D. Desta, H. Budak and H. Kara, *New perspectives on fractional Milne-type inequalities: Insights from twice-differentiable functions*, Univ. J. Math. Appl., **7**(1) (2024), 30-37. [CrossRef] [DergiPark] [Scopus]

[20] S. Kermausuor, *Simpson's type inequalities via the Katugampola fractional integrals for s-convex functions*, Kragujevac J. Math., **45**(5) (2021), 709-720. [CrossRef] [Scopus] [Web of Science]

[21] B.Z. Liu, *An inequality of Simpson type*, Proc. R. Soc. A, **461** (2005), 2155-2158. [CrossRef] [Scopus] [Web of Science]

[22] H. Lei, G. Hu, J. Nie and T. Du, *Generalized Simpson-type inequalities considering first derivatives through the k-Fractional Integrals*, IAENG Int. J. Appl. Math., **50**(3) (2020), 1-8. [Scopus]

[23] S. Rashid, A.O. Akdemir, F. Jarad, M.A. Noor and K.I. Noor, *Simpson's type integral inequalities for κ-fractional integrals and their applications*, AIMS Math., **4**(4) (2019), 1087-1100. [CrossRef] [Scopus] [Web of Science]

[24] M.Z. Sarıkaya H. Budak and S. Erden, *On new inequalities of Simpson's type for generalized convex functions*, Korean J. Math., **27**(2) (2019), 279-295. [CrossRef] [Web of Science]

[25] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204, Elsevier Sci. B.V., Amsterdam, 2006.

[26] C. Luo and T. Du, *Generalized Simpson type inequalities involving Riemann-Liouville fractional integrals and their applications*, Filomat, **34**(3) (2020), 751-760. [CrossRef] [Scopus] [Web of Science]

[27] S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, NewYork:Wiley, 1993.

[28] R. Gorenflo and F. Mainardi, *Fractional Calculus: Integral and Differential Equations of Fractional Order*, Springer Verlag, Wien (1997), 223-276. [arXiv]

[29] X. Hai and S.H. Wang *Simpson type inequalities for convex function based on the generalized fractional integrals*, Turkish J. Inequal., **5**(1) (2021), 1-15. [Web]

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