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Research Paper

A Subclass of Analytic Functions Involving Certain Mathieu-Type Series

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Abstract

In our present work, we first study certain Mathieu-type series and then define a new subclass of Pascu-type analytic functions. Also some inheriting results like the Fekete-Szegö functional, radius problems, a number of sufficient conditions and results related to partial sums are derived. Some new and known consequences of our main results are also given.

Key Words: Analytic functions; Starlike function; Pascu-type analytic functions; Janowski function; Mathieu-type series

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1. Introduction

We denote the class of analytic function having the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad (z \in \mathbb{D}), \tag{1}$$

by $\mathcal{A}$ in the open unit disk

$$\mathbb{D} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Next, for any two functions $g_1, g_2 \in \mathcal{A}$, the function g_1 is said to be subordinate to the function g_2, which is written as

$$g_1 \prec g_2,$$

if Schwarz function ϖ exist holomorphic in $\mathbb{D}$ with the following conditions

$$\varpi(0) = 0 \quad \text{and} \quad |\varpi(z)| < 1,$$

such that

$$g_1(z) = g_2(\varpi(z)) \qquad (z \in \mathbb{D}).$$

For two analytic functions f and g, the convolution (Hadamard product) of f and g is defined as:

$$f(z) * g(z) = \sum_{k=0}^{\infty} a_k b_k z^k.$$

Let $\mathcal{P}$ denote the well-known Carathéodory class of functions p, analytic in the open unit disk $\mathbb{D}$, which are normalized by

$$p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k, \tag{2}$$

such that

$$\Re\{p(z)\} > 0 \qquad (\forall\ z \in \mathbb{D}).$$

Furthermore, we denoted by $\mathcal{N}$ the class of all those well-known functions, which are univalent in $\mathbb{D}$. A function $f \in \mathcal{A}$ is said to be in $\mathcal{N}^*$, the class of starlike functions, if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \qquad (z \in \mathbb{D}).$$

Equivalently

$$\mathcal{N}^*(\varphi) = \left\{f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z)\right\}, \tag{3}$$

where

$$\varphi(z) = \frac{(1+z)}{(1-z)}.$$

By giving some specific values to $\varphi(z)$, then we can get some known subclasses of $\mathcal{N}$, some of them are listed as follows:

1. For $\varphi(z) = 1 + \sin z$, we have the functions class $\mathcal{N}^*(\varphi)$ of starlike functions associated with the sine functions (see [1]).

2. For $\varphi(z) = 1 + z - \frac{1}{3}z^3$, we have the functions class of starlike functions associated with the nephroid (see [2]).

3. For $\varphi(z) = e^z$, we have the functions class of starlike functions associated with the exponential functions (see [3]).

4. For $\varphi(z) = z + \sqrt{1+z^2}$, we have the functions class of starlike functions associated with the crescent shaped region (see [4]).

The classes defined above play an important role in the development of this filed. Many interesting properties of each of the defined functions classes have been studied from different viewpoints and prospectives. Also, it could be seen that, as time passed some new subclasses were introduced and studied see [5, 6, 7] by taking φ in (3) with some other specific type of functions.

The class $\mathcal{N}^*[M, N]$ of starlike functions associated with the Janwoski function can be defined as follows.

Definition 1. [8] A function f is called in the class $\mathcal{N}^*[M,N]$ if

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Mz}{1+Nz} \qquad (-1 \leq N < M \leq 1).$$

Or

$$\frac{zf'(z)}{f(z)} = \frac{(M+1)\,p(z)-(M-1)}{(N+1)\,p(z)-(N-1)} \qquad (-1 \leq N < M \leq 1). \tag{4}$$

This function $\mathcal{N}^*[M,N]$ of starlike functions was given by Janowski [8].

The following series (Mathieu-type series) is named after 'Emile Leonard Mathieu (1835–1890), who examined it in his monograph [9] on the elasticity of solid bodies.

$$T(i) = \sum_{k=1}^{\infty} \frac{2k}{(k^2+i^2)^2} \qquad (i>0).$$

The series $T(i)$ has a closed integral form and is given by (see [10])

$$T(i) = \frac{1}{i}\int_0^{\infty} \frac{t\sin(it)}{e^t-1}dt.$$

The Mathieu-type series is defined as follows (see [11]):

$$T(i;z) = \sum_{k=1}^{\infty} \frac{2k}{(k^2+i^2)^2} z^k \qquad (i>0, \quad |z|<1).$$

It was originally created for functions of real variables, but Bansal et al. [12] extended it to complex variables. Since $T(i;z) \notin \mathcal{H}$ so using following normalization, we have

$$\begin{aligned} T(i;z) &= \frac{(i^2+1)^2}{2}\sum_{k=1}^{\infty} \frac{2k}{(k^2+i^2)^2} z^k \\ &= z + \sum_{k=2}^{\infty} \frac{k(i^2+1)^2}{(k^2+i^2)^2} z^k, \end{aligned} \tag{5}$$

for some related work we refer the reader to see [13]-[15].

The theory of operators play a vital role in the development of Geometric Function Theory. Many new operators have been studied systematically from many different aspects and by means of these operators some useful subclasses have been defined and studied, see for example [16]. A number of integral and differential operators can be described in term of convolution. These operators are helpful in understanding the mathematical exploration and geometric configuration of analytic functions. The importance of convolution in the theory of operators may be understood by [17]-[22].

Using the Hadamard product in conjunction with (1) and (5), we introduce a new linear operator $H_k^i : \mathcal{A} \to \mathcal{A}$ as follows

$$H_k^i f(z) = f(z) * T(i;z) = z + \sum_{k=2}^{\infty} \frac{k(i^2+1)^2}{(k^2+i^2)^2} a_k z^k. \tag{6}$$

We now motivated by the above mentioned works and define the following function class of starlike functions involving the Janwoski functions.

Definition 2. A function $H_k^i f(z)$ of the form (6) is said to be in the class $\mathcal{N}_\lambda^{*m}(M,N)$ if and only if

$$\frac{z\left(H_k^i f(z)\right)'}{(1-\lambda)z+\lambda H_k^i f(z)} \prec \frac{1+Mz}{1+Nz} \qquad (z\in\mathbb{D}). \tag{7}$$

Or equivalently we can write the above subordination as follows:

$$\left|\frac{(1-\lambda)z+\lambda H_k^i f(z)-z\left(H_k^i f(z)\right)'}{Nz\left(H_k^i f(z)\right)'-M\left\{(1-\lambda)z+\lambda H_k^i f(z)\right\}}\right|<1.$$

By taking $\lambda=0$, we state the following class:

Definition 3. A function $H_k^i f(z)$ of the form (6) is said to be in the class $\mathcal{R}(M,N)$ if and only if

$$\left(H_k^i f(z)\right)' \prec \frac{1+Mz}{1+Nz} \qquad (z\in\mathbb{D}).$$

Or equivalently we can write the above subordination as follows:

$$\left|\frac{\left(H_k^i f(z)\right)'}{N\left(H_k^i f(z)\right)'-M}\right|<1.$$

Remark 1. If we let $\lambda=1$, in the above Definition, we have the function class defined and studied by [18].

The problems related to the coefficients of a function from the class $\mathcal{A}$ is a core of attractions for many mathematicians. One of the most important problem related to coefficients of the functions f is the Bieberbach conjecture which was solved by De-Branges, 70 years after its formulation. The Fekete-Szegö functional $\left|a_3-a_2^2\right|$ is also one of the important finding for the coefficients of the functions f. This functional is further generalized as $\left|a_3-\mu a_2^2\right|$. Fekete and Szegö gave sharp estimates of $\left|a_3-\mu a_2^2\right|$ for a real μ and $f\in\mathcal{N}$.

In Geometric Function Theory of Complex Analysis many authors have motivated by the problems related to coefficients. We have chosen to add some remarkable recent work on this subject (see for example [23], [24], [25] and [26]) on various functions classes of analytic and bi-univalent functions. Also one may attempt to produce the same results for a function class defined in different domains. Particularly, interested can also obtain the q-extension of the defined functions classes and results. Furthermore, the works presented in [23], [24], [25] and [26] can also be generalize by connecting it with some special type of series.

Here in this paper, we find the Fekete-Szegö functional $\left|a_3-\mu a_2^2\right|$ for our defined functions class $\mathcal{N}_\lambda^{*m}(M,N)$. Also we give a number of sufficient conditions for a function to be in our newly defined function class. Some other interesting results, like partial sums, desertion Theorems and radius problems are derived. Relevant connection to those other related works are also highlighted. To find the Fekete-Szegö functional, we need the following Lemma.

Lemma 1. ([20] and [21]) Let

$$p(z)=1+c_1z+c_2z^2+\ldots$$

be in the class $\mathcal{P}$. Then for any complex number υ

$$\left|c_2-\upsilon c_1^2\right| \leqq 2\max\{1,|1-2\upsilon|\}.$$

In particular, if υ is a real parameter, then

$$\left|c_2-\upsilon c_1^2\right| \leqq \begin{cases} -4\upsilon+2 & (\upsilon \leqq 0) \\ 2 & (0 \leqq \upsilon \leqq 1) \\ 4\upsilon-2 & (\upsilon \geqq 1)\,. \end{cases} \tag{8}$$

2. Main Results

Our first result is related to Fekete-Szegö functional.

Theorem 1. *Let the function $H_k^i f(z)$ given by (6) be in the class $\mathcal{N}_\lambda^{*m}(M,N)$. Then for a complex number μ,*

$$\left|a_3-\mu a_2^2\right| \leq \frac{(M-N)\left(i^2+9\right)^2}{6\left(i^2+1\right)^2} \max\left\{1, \left|\frac{\rho_1(M,N,k)-3\mu(M-N)\left(i^2+4\right)^4(3-\lambda)}{4(\lambda-2)^2\left(i^2+9\right)^2\left(i^2+1\right)^2}\right|\right\}, \tag{9}$$

where

$$\rho_1(M,N,k)=4(\lambda-2)(M-3N+(N+1)\lambda-2)\left(i^2+9\right)^2\left(i^2+1\right)^2.$$

Furthermore, for a real parameter μ,

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \left(\frac{(M-N)}{6(i^2+1)^2(3-\lambda)}\right)\left(\frac{\rho_2(M,N,k,\lambda)-3\mu(M-N)\left(i^2+4\right)^4(3-\lambda)}{3(\lambda-2)^2(i^2+1)^2}\right) & (\mu<\sigma_1) \\ \left(\frac{(M-N)\left(i^2+9\right)^2}{3(i^2+1)^2(3-\lambda)}\right) & (\sigma_1 \leq \mu \leq \sigma_2) \\ \left(\frac{(N-M)}{6(i^2+1)^2(3-\lambda)}\right)\left(\frac{\rho_2(M,N,k,\lambda)-3\mu(M-N)\left(i^2+4\right)^4(3-\lambda)}{3(\lambda-2)^2(i^2+1)^2}\right) & (\mu>\sigma_2)\,, \end{cases} \tag{10}$$

where

$$\rho_2(M,N,k)=4((N+2)\lambda+M-3N-4)(\lambda-2)\left(i^2+9\right)^2\left(i^2+1\right)^2, \tag{11}$$

$$\sigma_1=\frac{4(M-3N+N\lambda+\lambda-2)(\lambda-2)\left(i^2+9\right)^2\left(i^2+1\right)^2}{3(M-N)(3-\lambda)\left(i^2+4\right)^4}$$

and

$$\sigma_2=\frac{4((N+3)\lambda+M-3N-6)(\lambda-2)\left(i^2+9\right)^2\left(i^2+1\right)^2}{3(M-N)(3-\lambda)\left(i^2+4\right)^4}.$$

Proof. We begin by showing that the inequalities (9) and (10) hold true for $H_k^i f(z) \in \mathcal{N}_\lambda^{*m}(M,N)$. Since $H_k^i f(z) \in \mathcal{N}_\lambda^{*m}(M,N)$, therefore, we have the following subordination:

$$\frac{z\left(H_k^i f(z)\right)'}{(1-\lambda)z + \lambda H_k^i f(z)} \prec \frac{1+Mz}{1+Nz}. \tag{12}$$

The above subordination can also be written as:

$$\frac{z\left(H_k^i f(z)\right)'}{(1-\lambda)z + \lambda H_k^i f(z)} = \frac{(M+1)p(z)-(M-1)}{(N+1)p(z)-(N-1)} \qquad (-1 \leq N < M \leq 1).$$

Now let us consider

$$\frac{(M+1)p(z)-(M-1)}{(N+1)p(z)-(N-1)} = 1 + \frac{1}{2}(M-N)p_1 z + \left\{\frac{1}{2}(M-N)p_2 - \frac{1}{4}(M-N)(N+1)p_1^2\right\}z^2 + \ldots \tag{13}$$

Also

$$\frac{z\left(H_k^i f(z)\right)'}{(1-\lambda)z + \lambda H_k^i f(z)} = 1 + \frac{2(2-\lambda)\left(i^2+1\right)^2}{\left(i^2+4\right)^2}a_2 z + \left\{\frac{3(3-\lambda)\left(i^2+1\right)^2}{\left(i^2+9\right)^2}a_3 - \frac{4(\lambda-2)\left(i^2+1\right)^4}{\left(i^2+4\right)^4}a_2^2\right\}z^2 + \ldots \tag{14}$$

We find from the equations (14) and (13) that

$$a_2 = \frac{(M-N)\left(i^2+4\right)^2}{4(2-\lambda)\left(i^2+1\right)^2}p_1 \tag{15}$$

and

$$a_3 = \frac{(M-N)\left(i^2+9\right)^2}{6(3-\lambda)\left(i^2+1\right)^2}\left[\left((N+1)(\lambda-2)+M-N\right)\frac{p_1^2}{2} + p_2\right]. \tag{16}$$

Thus, clearly, we find that

$$\left|a_3 - \mu a_2^2\right| = \frac{(M-N)\left(i^2+9\right)^2}{6(3-\lambda)\left(i^2+1\right)^2}\left|p_2 - \Omega p_1^2\right|, \tag{17}$$

where

$$\Omega = \frac{3\mu(M-N)\left(i^2+4\right)^4(3-\lambda)}{8(2-\lambda)^2\left(i^2+9\right)^2\left(i^2+1\right)^2} - \frac{(N+1)(\lambda-2)+(M-N)}{2(\lambda-2)}.$$

Finally, by applying Lemma 1 in conjunction with (17), we obtain the result asserted by Theorem 1. □

Theorem 2. *Let $H_k^i f(z) \in \mathcal{N}_\lambda^{*m}(M,N)$ be given by (1). Then*

$$\sum_{k=2}^{\infty} \frac{k\left(i^2+1\right)^2}{\left(k^2+i^2\right)^2}\left(k-1+|Nk-M\lambda|\right)|a_k| \leq M-N. \tag{18}$$

Proof. Let $H_k^i f(z) \in \mathcal{N}_\lambda^{*m}(M,N)$. Then, (7) can be put in the form of Schwarz function $\varpi(z)$ as

$$\frac{z\left(H_k^i f(z)\right)'}{(1-\lambda)z+\lambda H_k^i f(z)} = \frac{1+M\varpi(z)}{1+N\varpi(z)} \qquad (z\in\mathbb{D}). \tag{19}$$

Or equivalently

$$\left|\frac{(1-\lambda)z+\lambda Q(k,i)f(z)-z(Q(k,i)f)'(z)}{Nz\left(H_k^i f(z)\right)'-M\left\{(1-\lambda)z+\lambda H_k^i f(z)\right\}}\right|<1.$$

Consider

$$\begin{aligned}\left|\frac{(1-\lambda)z+\lambda H_k^i f(z)-z\left(H_k^i f(z)\right)'}{Nz\left(H_k^i f(z)\right)'-M\left\{(1-\lambda)z+\lambda H_k^i f(z)\right\}}\right| &= \left|\frac{\sum_{k=2}^{\infty}\frac{k\left(i^2+1\right)^2}{(i^2+k^2)^2}[k-1]a_k z^k}{(M-N)z+\sum_{k=2}^{\infty}\frac{k(i^2+1)^2}{(i^2+k^2)^2}[Nk-M\lambda]a_k z^k}\right| \\ &\le \frac{\sum_{k=2}^{\infty}\frac{k\left(i^2+1\right)^2}{(i^2+k^2)^2}[k-1]|a_k|}{(M-N)+\sum_{k=2}^{\infty}\left(\frac{k(i^2+1)^2}{(i^2+k^2)^2}|Nk-M\lambda|\right)|a_k|} \\ &< 1,\end{aligned}$$

after simple computation we get the required inequality (34). ☐

Example 1. For the function

$$f(z) = z+\sum_{k=2}^{\infty}\frac{\left(k^2+i^2\right)^2(M-N)}{k\left(i^2+1\right)^2(k-1+|Nk-M\lambda|)}\upsilon_k z^k \qquad (z\in\mathbb{D}),$$

such that $\sum_{k=2}^{\infty}\upsilon_k=1$, we have

$$\begin{aligned}\sum_{k=2}^{\infty}\frac{k\left(i^2+1\right)^2}{\left(k^2+i^2\right)^2}(k-1+|Nk-M\lambda|)|a_k| &= \sum_{k=2}^{\infty}\frac{k\left(i^2+1\right)^2}{\left(k^2+i^2\right)^2}(k-1+|Nk-M\lambda|) \\ &\times\left(\frac{\left(k^2+i^2\right)^2(M-N)}{k\left(i^2+1\right)^2(k-1+|Nk-M\lambda|)}\upsilon_k\right) \\ &= (M-N)\sum_{k=2}^{\infty}\upsilon_k = (M-N).\end{aligned}$$

Corollary 1. Let $H_k^i f(z) \in \mathcal{N}_\lambda^{*m}(M,N)$ and be of the form (1). Then

$$|a_k| \le \frac{\left(k^2+i^2\right)^2(M-N)}{k\left(i^2+1\right)^2(k-1+|Nk-M\lambda|)} \qquad (k\ge 2). \tag{20}$$

Proof. The proof is quite straightforward, left for the reader. ☐

Theorem 3. *Let $H_k^i f(z) \in \mathcal{N}_\lambda^{*m}(M,N)$ and be of the form (1). Then*

$$r-\frac{\left(4+i^2\right)^2(M-N)}{2\left(i^2+1\right)^2(1+|2N-M\lambda|)}r^2 \le |f(z)| \le r+\frac{\left(4+i^2\right)^2(M-N)}{2\left(i^2+1\right)^2(1+|2N-M\lambda|)}r^2. \tag{21}$$

Proof. Consider

$$|f(z)| = \left| z + \sum_{k=2}^{\infty} a_k z^k \right| \leq |z| + \sum_{k=2}^{\infty} |a_k| \, |z|^k$$
$$= i + \sum_{k=2}^{\infty} |a_k| \, |i|^k ,$$

since for $|z| = r < 1$ we have $r^k < r^2$ for $k \geq 2$ and

$$|f(z)| \leq r + r^2 \sum_{k=2}^{\infty} |a_k| .$$

Comparably

$$|f(z)| \geq r - r^2 \sum_{k=2}^{\infty} |a_k| .$$

Now from (34) implies that

$$\sum_{k=2}^{\infty} \frac{k\left(i^2+1\right)^2}{\left(k^2+i^2\right)^2} \left(k - 1 + |Nk - \lambda M|\right) |a_k| \leq M - N.$$

But

$$\sum_{k=2}^{\infty} \frac{2\left(i^2+1\right)^2}{\left(4+i^2\right)^2} \left(1 + |2N - \lambda M|\right) |a_k| \leq \sum_{k=2}^{\infty} \frac{k\left(i^2+1\right)^2}{\left(k^2+i^2\right)^2} \left(k - 1 + |Nk - \lambda M|\right) |a_k| \leq M - N,$$

which gives

$$\sum_{k=2}^{\infty} |a_k| \leq \frac{\left(4+i^2\right)^2 (M-N)}{2\left(i^2+1\right)^2 \left(1 + |2N - \lambda M|\right)}.$$

□

Theorem 4. *Let $H_k^i f(z) \in \mathcal{N}_\lambda^{*m}(M,N)$ and be of the form (1). Then*

$$r - \frac{\left(4+i^2\right)^2 (M-N)}{\left(i^2+1\right)^2 \left(1 + |2N - \lambda M|\right)} r^2 \leq \left|f'(z)\right| \leq r + \frac{\left(4+i^2\right)^2 (M-N)}{\left(i^2+1\right)^2 \left(1 + |2N - \lambda M|\right)} r^2. \tag{22}$$

Proof. The proof is quite similar as Theorem 3, so omitted. □

Theorem 5. *Let $f_i \in \mathcal{N}_\lambda^{*m}(M,N)$ and have of the from*

$$f_i(z) = z + \sum_{k=2}^{\infty} a_{i,k} z^k \qquad (i = 1, 2, 3, ..., k). \tag{23}$$

Then $H \in \mathcal{N}_\lambda^{*m}(M,N)$, where

$$H(z) = \sum_{i=1}^{k} c_i f_i(z) \qquad \text{with} \qquad \sum_{i=1}^{k} |c_i| = 1. \tag{24}$$

Proof. From Theorem 2, we can write

$$\sum_{k=2}^{\infty} \frac{k\left(i^2+1\right)^2}{\left(k^2+i^2\right)^2}\left(k-1+|Nk-\lambda M|\right)a_k \leq M-N.$$

Also,

$$\begin{aligned} H(z) &= \sum_{i=1}^{k} c_i\left(z+\sum_{k=2}^{\infty} a_{i,k}z^k\right) \\ &= z+\sum_{k=2}^{\infty}\left(\sum_{i=1}^{k} c_i a_{i,k}\right)z^k, \end{aligned}$$

therefore

$$\begin{aligned} \sum_{k=2}^{\infty} \frac{k\left(i^2+1\right)^2}{\left(k^2+i^2\right)^2}\left(k-1+|Nk-M\lambda|\right)\left|\sum_{i=1}^{k} c_i a_{i,k}\right| &= \sum_{i=1}^{k}\left[\sum_{k=2}^{\infty} \frac{k\left(i^2+1\right)^2}{\left(k^2+i^2\right)^2}\left(k-1+|Nk-M\lambda|\right)|a_{i,k}|\right]|c_i| \\ &\leq \sum_{i=1}^{k}(M-N)|c_i| = (M-N)\sum_{i=1}^{k}|c_i| = M-N, \end{aligned}$$

thus $H(z) \in \mathcal{N}_{\lambda}^{*m}(M,N)$. □

Remark 2. If we put $\lambda = 1$, in the above theorem, we will arrived at the result that was already proved in [18].

Theorem 6. *Let $f_i \in \mathcal{N}_{\lambda}^{*m}(M,N)$, for $i = 1, 2, ..., j$. Then the arithmetic mean h of f_i is given by*

$$h(z) = \frac{1}{j}\sum_{k=1}^{j} f_i(z), \tag{25}$$

and also belong to class $\mathcal{N}_{\lambda}^{*m}(M,N)$.

Proof. From (25), we can write

$$\begin{aligned} h(z) = \frac{1}{j}\sum_{k=1}^{j} f_i(z) &= \frac{1}{j}\sum_{k=1}^{j}\left(z+\sum_{k=2}^{\infty} a_{j,k}z^k\right) \\ &= z+\sum_{k=2}^{\infty}\left(\frac{1}{j}\sum_{k=1}^{j} a_{j,k}\right)z^k, \end{aligned}$$

to demonstrate that $h(z)$ belong to $\mathcal{N}_{\lambda}^{*m}(M,N)$, it's enough to show that

$$\sum_{k=2}^{\infty} \frac{k\left(i^2+1\right)^2}{\left(k^2+i^2\right)^2}\left(k-1+|Nk-M\lambda|\right)\left|\frac{1}{j}\sum_{k=1}^{j} a_{j,k}\right| \leq M-N.$$

Consider

$$\sum_{k=2}^{\infty} \frac{k\left(i^2+1\right)^2}{\left(k^2+i^2\right)^2}\left(k-1+|Nk-M\lambda|\right)\left|\frac{1}{j}\sum_{k=1}^{j} a_{j,k}\right| = \frac{1}{j}\sum_{k=1}^{j}\left(\sum_{k=2}^{\infty} \frac{k\left(i^2+1\right)^2}{\left(k^2+i^2\right)^2}\left(k-1+|Nk-M\lambda|\right)|a_{j,k}|\right)$$

$$\leq \frac{1}{j}\sum_{k=1}^{j}(M-N)=(M-N),$$

this show that $M(z)$ belong to $\mathcal{N}_{\lambda}^{*m}(M,N)$. $\square$

Theorem 7. *Let $f\in\mathcal{N}_{\lambda}^{*m}(M,N)$, then f is in class of starlike functions of order β $(0\leq\beta<1)$ for $|z|<i^{*}$, where*

$$i^{*}=\left(\frac{(1-\beta)\,k\left(i^{2}+1\right)^{2}}{(k-\beta)\left(k^{2}+i^{2}\right)^{2}}\frac{(k-1+|Nk-M\lambda|)}{M-N}\right)^{\frac{1}{k-1}}.$$

Proof. Let $H_{k}^{i}f(z)\in\mathcal{N}_{\lambda}^{*m}(M,N)$. To prove f is in class of starlike functions of order β, it's enough to show that

$$\left|\frac{zf'(z)-f(z)}{zf'(z)+(1-2\beta)f(z)}\right|<1.$$

Using 1 along with some basic math yields

$$\sum_{k=2}^{\infty}\left(\frac{k-\beta}{1-\beta}\right)|a_{k}|\,|z|^{k-1}<1. \tag{26}$$

Since $H_{k}^{i}f(z)\in\mathcal{N}_{\lambda}^{*m}(M,N)$, from (18) we have

$$\sum_{k=2}^{\infty}\frac{k\left(i^{2}+1\right)^{2}}{\left(k^{2}+i^{2}\right)^{2}}\frac{(k-1+|Nk-M\lambda|)}{M-N}|a_{k}|<1. \tag{27}$$

Inequality (26) will holds true if the following holds true:

$$\sum_{k=2}^{\infty}\left(\frac{k-\beta}{1-\beta}\right)|a_{k}|\,|z|^{k-1}<\sum_{k=2}^{\infty}\frac{k\left(i^{2}+1\right)^{2}}{\left(k^{2}+i^{2}\right)^{2}}\frac{(k-1+|Nk-\lambda M|)}{M-N}|a_{k}|,$$

which implies that

$$|z|^{k-1}<\left(\frac{(1-\beta)\,k\left(i^{2}+1\right)^{2}}{(k-\beta)\left(k^{2}+i^{2}\right)^{2}}\frac{(k-1+|Nk-\lambda M|)}{M-N}\right),$$

thus we get required result. $\square$

Theorem 8. *Let $f_{1}(z)=z$ and*

$$f_{k}(z)=z-\frac{\left(k^{2}+i^{2}\right)^{2}(M-N)}{k\left(i^{2}+1\right)^{2}(k-1+|Nk-\lambda M|)}z^{k}\qquad(z\in\mathbb{D},\quad k\geq 2).$$

Then $H_{k}^{i}f(z)\in\mathcal{N}_{\lambda}^{*m}(M,N)$ if and only if $Q(k,i)f$ can be expressed in the form

$$H_{k}^{i}f(z)=\sum_{k=1}^{\infty}\delta_{k}f_{k}(z)\qquad(\delta_{k}\geq 0), \tag{28}$$

and

$$\sum_{k=1}^{\infty}\delta_{k}=1.$$

Proof. From (28) , we can easily write

$$\begin{aligned} H_k^i f(z) &= \sum_{k=1}^{\infty} \delta_k f_k(z) \\ &= z + \sum_{k=2}^{\infty} \delta_k \frac{\left(k^2+i^2\right)^2 (M-N)}{k\left(i^2+1\right)^2 (k-1+|Nk-M|)} z^k, \end{aligned}$$

then from Theorem 2, we can write

$$\begin{aligned} \sum_{k=2}^{\infty} \frac{k\left(i^2+1\right)^2}{\left(k^2+i^2\right)^2} (k-1+|Nk-M\lambda|) \frac{\left(k^2+i^2\right)^2 \delta_k}{k\left(i^2+1\right)^2 (k-1+|Nk-M\lambda|)} &= (M-N) \sum_{k=2}^{\infty} \delta_k \\ &= (M-N)(1-\delta_1) \\ &\leq M-N. \end{aligned}$$

Thus by Theorem 2, $H_k^i f(z) \in \mathcal{N}_\lambda^{*m}(M,N)$. Conversely, let $H_k^i f(z) \in \mathcal{N}_\lambda^{*m}(M,N)$ since the Theorem 2, we have

$$|a_k| \leq \frac{\left(k^2+i^2\right)^2 (M-N)}{k\left(i^2+1\right)^2 (k-1+|Nk-M\lambda|)} \qquad (k \geq 2),$$

we set

$$\delta_k = \frac{k\left(i^2+1\right)^2 (k-1+|Nk-M\lambda|)}{\left(k^2+i^2\right)^2 (M-N)} |a_k|, \quad (k \geq 2),$$

and

$$\delta_1 = 1 - \sum_{k=2}^{\infty} \delta_k,$$

so it follows that

$$f(z) = \sum_{k=1}^{\infty} \delta_k f_k(z).$$

Hence proof is completed. □

Remark 3. If we put $\lambda = 1$, in the above Theorems 1-8 , we will arrived at the result that was already proved in [18].

3. Partial Sum

In this section, we will examine the ratio of a function of the form (1) to its sequence of partial sums

$$f_j(z) = z + \sum_{k=2}^{j} a_k z^k,$$

when the coefficients of f are sufficiently small to satisfy the condition (18). We will determine sharp lower bounds for

$$\Re\left(\frac{f(z)}{f_j(z)}\right), \quad \Re\left(\frac{f_j(z)}{f(z)}\right), \quad \Re\left(\frac{f'(z)}{f_j'(z)}\right) \quad \text{and} \quad \Re\left(\frac{f_j'(z)}{f'(z)}\right).$$

Theorem 9. *If f of the form (1) satisfies condition (18), then*

$$\Re\left(\frac{f(z)}{f_j(z)}\right) \geq 1-\frac{1}{\vartheta_{j+1}} \qquad (\forall z \in \mathbb{D}), \tag{29}$$

and

$$\Re\left(\frac{f_j(z)}{f(z)}\right) \geq \frac{\vartheta_{j+1}}{1+\vartheta_{j+1}} \qquad (\forall z \in \mathbb{D}), \tag{30}$$

where

$$\vartheta_j = \frac{k\left(i^2+1\right)^2 (k-1+|Nk-\lambda M|)}{\left(k^2+i^2\right)^2 (M-N)}. \tag{31}$$

Proof. For proving the inequality in (29), we suppose that:

$$\vartheta_{j+1}\left[\frac{f(z)}{f_j(z)}-\left(1-\frac{1}{\vartheta_{j+1}}\right)\right] = \frac{1+\sum\limits_{k=2}^{j} a_k z^{k-1}+\vartheta_{j+1}\sum\limits_{k=j+1}^{\infty} a_k z^{k-1}}{1+\sum\limits_{k=2}^{j} a_k z^{k-1}}$$
$$= \frac{1+\psi_1(z)}{1+\psi_2(z)}.$$

We now set:

$$\frac{1+\psi_1(z)}{1+\psi_2(z)} = \frac{1+\varpi(z)}{1-\varpi(z)}.$$

After some straightforward simplification, we have that:

$$\varpi(z) = \frac{\psi_1(z)-\psi_2(z)}{2+\psi_1(z)+\psi_2(z)}.$$

Thus, clearly, we find that:

$$\varpi(z) = \frac{\vartheta_{j+1}\sum\limits_{k=j+1}^{\infty} a_k z^{k-1}}{2+2\sum\limits_{k=2}^{j} a_k z^{k-1}+\vartheta_{j+1}\sum\limits_{k=j+1}^{\infty} a_k z^{k-1}}.$$

By using the triangular inequalities along with $|z|<1$, we can get the following easily:

$$|\varpi(z)| \leq \frac{\vartheta_{j+1}\sum\limits_{k=j+1}^{\infty} |a_k|}{2-2\sum\limits_{k=2}^{j} |a_k|-\vartheta_{j+1}\sum\limits_{k=j+1}^{\infty} |a_k|}.$$

Now $|\varpi(z)| \leq 1$, if and only if

$$2\vartheta_{j+1}\sum_{k=j+1}^{\infty} |a_k| \leq 2-2\sum_{k=2}^{j} |a_k|,$$

or, equivalently

$$\sum_{k=2}^{j} |a_k| + \vartheta_{j+1} \sum_{k=j+1}^{\infty} |a_k| \leq 1. \tag{32}$$

Finally, in order to prove inequality in (29), we must show that the left hand side of (32) is bounded above by the sum given by:

$$\sum_{k=2}^{\infty} \vartheta_k |a_k|,$$

we have

$$\sum_{k=2}^{j} (\vartheta_k - 1) |a_k| + \sum_{k=j+1}^{\infty} (\vartheta_k - \vartheta_{j+1}) |a_k| \geq 0. \tag{33}$$

By using (33), we see that the proof of inequality in (29) is completed.

Next in order to prove the inequality (30), we set:

$$\begin{aligned} (1+\vartheta_{j+1}) \left(\frac{f_j(z)}{f(z)} - \frac{\vartheta_{j+1}}{1+\vartheta_{j+1}} \right) &= \frac{1 + \sum_{k=2}^{j} a_k z^{k-1} - \vartheta_{j+1} \sum_{k=j+1}^{\infty} a_k z^{k-1}}{1 + \sum_{k=2}^{\infty} a_k z^{k-1}} \\ &= \frac{1+\varpi(z)}{1-\varpi(z)}, \end{aligned}$$

we can write

$$|\varpi(z)| \leq \frac{(1+\vartheta_{j+1}) \sum_{k=j+1}^{\infty} |a_k|}{2 - 2\sum_{k=2}^{j} |a_k| - (\vartheta_{j+1} - 1) \sum_{k=j+1}^{\infty} |a_k|} \leq 1. \tag{34}$$

This last inequality is equivalent to

$$\sum_{k=2}^{j} |a_k| + \vartheta_{j+1} \sum_{k=j+1}^{\infty} |a_k| \leq 1. \tag{35}$$

Finally we can see that the left hand side of the inequality in (35) is bounded above by the follwoing sum:

$$\sum_{k=2}^{\infty} \vartheta_k |a_k|,$$

so we have completed the proof of the assertion (30) . Which completes the proof of Theorem 9. □

We next turn to ratios involving derivatives.

Theorem 10. *If f of the form (1) satisfies condition (18), then*

$$\Re\left(\frac{f'(z)}{f_j'(z)}\right) \geq 1 - \frac{j+1}{\vartheta_{j+1}} \qquad (\forall z \in \mathbb{D}), \tag{36}$$

and

$$\Re\left(\frac{f_j'(z)}{f'(z)}\right) \geq \frac{\vartheta_{j+1}}{\vartheta_{j+1} + j + 1} \qquad (\forall z \in \mathbb{D}), \tag{37}$$

where ϑ_j is given by (31).

Proof. The proof of Theorem 10 is similar to that of Theorem 9, we here choose to omit the analogous details. □

Remark 4. If we put $\lambda = 1$, in the above Theorems in this section, we will arrived at the result that was already proved in [18].

Remark 5. If we put $\lambda = 0$, one can easily state the result discussed in the paper for the class $\mathcal{R}(M, N)$ given in Definition 3.

4. Conclusion

The theory of operators play a vital role in the development of Geometric Function Theory. Many new operators have been studied systematically from many different aspects and by means of these operators some useful subclasses have been defined and studied, see for example [16] and [19] see also [27, 28]. A number of integral and differential operators can be described in term of convolution. These operators are helpful to understood the mathematical exploration and geometric configuration of analytic functions.

In our present work, we have first studied certain Mathieu-type series and then have defined a new subclass of Pascu-type analytic functions in the Janwoski domain. Also some inheriting results like the Fekete-Szegö functional, radius problems, a number of sufficient conditions and results related to partial sums have been derived for our defined function classes.

Declarations

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