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Research Paper

Some Discrete Inequalities for Convex Functions Defined on Linear Spaces

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Abstract

The following inequality is the well-known Hermite-Hadamard integral inequality for convex functions defined on a segment

$$[a,b] := \{(1-t)\,a + tb, t \in [0,1]\}$$

with a, b vectors in a linear space X,

$$f\left(\frac{a+b}{2}\right) \leq \int_0^1 f\left[(1-t)\,a + tb\right] dt \leq \frac{f(a)+f(b)}{2}.$$

In this paper we provide some discrete inequalities related to the Hermite-Hadamard result for convex functions defined on convex subsets in a linear space. Applications for norms and univariate real functions with an example for the logarithm, are also given.

Key Words: Convex functions, Linear spaces, Jensen's inequality, Hermite-Hadamard inequality, Norm inequalities.
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1. Introduction

Let X be a real linear space, $a,\ b \in X,\ a \neq b$ and let $[a,b] := \{(1-\lambda)\,a + \lambda b,\ \lambda \in [0,1]\}$ be the *segment* generated by a and b. We consider the function $f : [a,b] \to \mathbb{R}$ and the attached function $g\,(a,b) : [0,1] \to \mathbb{R}$, $g\,(a,b)\,(t) := f\left[(1-t)\,a + tb\right]$, $t \in [0,1]$.

It is well known that f is convex on $[a,b]$ iff $g\,(a,b)$ is convex on $[0,1]$, and the following lateral derivatives exist and satisfy

(i) $g'_{\pm}\,(a,b)\,(s) = \left(\triangledown_{\pm} f\left[(1-s)\,a + sb\right]\right)(b-a),\ s \in [0,1)$
(ii) $g'_{+}\,(a,b)\,(0) = \left(\triangledown_{+} f\,(a)\right)(b-a)$
(iii) $g'_{-}\,(a,b)\,(1) = \left(\triangledown_{-} f\,(b)\right)(b-a)$

where $(\nabla_{\pm} f(x))(y)$ are the *Gâteaux lateral derivatives,* we recall that

$$\begin{aligned}(\nabla_{+} f(x))(y) &: = \lim_{h\to 0+}\left[\frac{f(x+hy)-f(x)}{h}\right],\\ (\nabla_{-} f(x))(y) &: = \lim_{k\to 0-}\left[\frac{f(x+ky)-f(x)}{k}\right],\ \ x,\ y\in X.\end{aligned}$$

The following inequality is the well-known Hermite-Hadamard integral inequality for convex functions defined on a segment $[a,b]\subset X$:

$$f\left(\frac{a+b}{2}\right)\le\int_0^1 f[(1-t)a+tb]\,dt\le\frac{f(a)+f(b)}{2}, \tag{HH}$$

which easily follows by the classical Hermite-Hadamard inequality for the convex function $g(a,b):[0,1]\to\mathbb{R}$

$$g(a,b)\left(\frac{1}{2}\right)\le\int_0^1 g(a,b)(t)\,dt\le\frac{g(a,b)(0)+g(a,b)(1)}{2}.$$

For other related results see the monograph on line [1]. For some Hermite-Hadamard type inequalities see [2], [3], [4] and the references therein.

We have the following result [5] related to the first Hermite-Hadamard inequality in (HH):

Theorem 1. *Let X be a linear space, $a,b\in X$, $a\neq b$ and $f:[a,b]\subset X\to\mathbb{R}$ be a convex function on the segment $[a,b]$. Then for any $s\in(0,1)$ one has the inequality*

$$\begin{aligned}&\frac{1}{2}\left[(1-s)^2(\nabla_{+} f[(1-s)a+sb])(b-a)-s^2(\nabla_{-} f[(1-s)a+sb])(b-a)\right] \\ &\le\int_0^1 f[(1-t)a+tb]\,dt-f[(1-s)a+sb] \\ &\le\frac{1}{2}\left[(1-s)^2(\nabla_{-} f(b))(b-a)-s^2(\nabla_{+} f(a))(b-a)\right].\end{aligned} \tag{1}$$

The constant $\frac{1}{2}$ is sharp in both inequalities. The second inequality also holds for $s=0$ or $s=1$.

If $f:[a,b]\to\mathbb{R}$ is as in Theorem 1 and Gâteaux differentiable in $c:=(1-\lambda)a+\lambda b$, $\lambda\in(0,1)$ along the direction $b-a$, then we have the inequality:

$$\left(\frac{1}{2}-\lambda\right)(\nabla f(c))(b-a)\le\int_0^1 f[(1-t)a+tb]\,dt-f(c). \tag{2}$$

If f is as in Theorem 1, then

$$\begin{aligned}0&\le\frac{1}{8}\left[\nabla_{+} f\left(\frac{a+b}{2}\right)(b-a)-\nabla_{-} f\left(\frac{a+b}{2}\right)(b-a)\right] \\ &\le\int_0^1 f[(1-t)a+tb]\,dt-f\left(\frac{a+b}{2}\right) \\ &\le\frac{1}{8}\left[(\nabla_{-} f(b))(b-a)-(\nabla_{+} f(a))(b-a)\right].\end{aligned} \tag{3}$$

The constant $\frac{1}{8}$ is sharp in both inequalities.

Also we have the following result [6] related to the second Hermite-Hadamard inequality in (HH):

Theorem 2. *Let X be a linear space, $a, b \in X$, $a \neq b$ and $f : [a,b] \subset X \to \mathbb{R}$ be a convex function on the segment $[a,b]$. Then for any $s \in (0,1)$ one has the inequality*

$$\begin{aligned}
&\frac{1}{2}\left[(1-s)^2\left(\triangledown_+ f\left[(1-s)a+sb\right]\right)(b-a) - s^2\left(\triangledown_- f\left[(1-s)a+sb\right]\right)(b-a)\right] \\
&\leq (1-s) f(a) + s f(b) - \int_0^1 f\left[(1-t)a+tb\right] dt \\
&\leq \frac{1}{2}\left[(1-s)^2\left(\triangledown_- f(b)\right)(b-a) - s^2\left(\triangledown_+ f(a)\right)(b-a)\right].
\end{aligned} \tag{4}$$

The constant $\frac{1}{2}$ is sharp in both inequalities. The second inequality also holds for $s=0$ or $s=1$.

If $f : [a,b] \to \mathbb{R}$ is as in Theorem 2 and Gâteaux differentiable in $c := (1-\lambda)a + \lambda b$, $\lambda \in (0,1)$ along the direction $b-a$, then we have the inequality:

$$\left(\frac{1}{2}-\lambda\right)\left(\triangledown f(c)\right)(b-a) \leq (1-\lambda) f(a) + \lambda f(b) - \int_0^1 f\left[(1-t)a+tb\right] dt. \tag{5}$$

If f is as in Theorem 2, then

$$\begin{aligned}
0 &\leq \frac{1}{8}\left[\triangledown_+ f\left(\frac{a+b}{2}\right)(b-a) - \triangledown_- f\left(\frac{a+b}{2}\right)(b-a)\right] \\
&\leq \frac{f(a)+f(b)}{2} - \int_0^1 f\left[(1-t)a+tb\right] dt \\
&\leq \frac{1}{8}\left[\left(\triangledown_- f(b)\right)(b-a) - \left(\triangledown_+ f(a)\right)(b-a)\right].
\end{aligned} \tag{6}$$

The constant $\frac{1}{8}$ is sharp in both inequalities.

2. The Results

Let $f : C \subset X \to \mathbb{R}$ be a convex function on C. We define the function $F_f : C \times C \to \mathbb{R}$ by

$$F_f(x,y) := \int_0^1 f\left((1-t)x+ty\right) dt. \tag{7}$$

Theorem 3. *Let $f : C \subset X \to \mathbb{R}$ be a convex function on C. Then the function F_f is convex on $C \times C$ and if $x_i,\ y_i \in C$ and $p_i \geq 0$ for $i = 1, ..., n$ with $\sum_{i=1}^n p_i = 1$, then we have the inequalities*

$$\begin{aligned}
\sum_{i=1}^n p_i \int_0^1 f\left((1-t)x_i+ty_i\right) dt &\geq \int_0^1 f\left((1-t)\sum_{i=1}^n p_i x_i + t\sum_{i=1}^n p_i y_i\right) dt \\
&\geq f\left(\sum_{i=1}^n p_i\left(\frac{x_i+y_i}{2}\right)\right),
\end{aligned} \tag{8}$$

$$\begin{aligned}
\sum_{i=1}^n p_i\left(\frac{f(x_i)+f(y_i)}{2}\right) &\geq \sum_{i=1}^n p_i \int_0^1 f\left((1-t)x_i+ty_i\right) dt \\
&\geq \sum_{i=1}^n p_i f\left(\frac{x_i+y_i}{2}\right) \geq f\left(\sum_{i=1}^n p_i\left(\frac{x_i+y_i}{2}\right)\right)
\end{aligned} \tag{9}$$

and

$$\sum_{i=1}^{n} p_i \left(\frac{f(x_i) + f(y_i)}{2} \right) \geq \frac{1}{2} \left[f\left(\sum_{i=1}^{n} p_i x_i \right) + f\left(\sum_{i=1}^{n} p_i y_i \right) \right] \tag{10}$$
$$\geq \int_0^1 f\left((1-t) \sum_{i=1}^{n} p_i x_i + t \sum_{i=1}^{n} p_i y_i \right) dt.$$

Proof. Let $(x,y), (u,v) \in C \times C$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. Then

$$\begin{aligned} F_f(\alpha(x,y) + \beta(u,v)) &= F_f(\alpha x + \beta u, \alpha y + \beta v) \\ &= \int_0^1 f((1-t)(\alpha x + \beta u) + t(\alpha y + \beta v)) \, dt \\ &= \int_0^1 f(\alpha[(1-t)x + ty] + \beta[(1-t)u + tv]) \, dt \\ &\leq \int_0^1 [\alpha f((1-t)x + ty) + \beta f((1-t)u + tv)] \, dt \\ &= \alpha \int_0^1 f((1-t)x + ty) \, dt + \beta \int_0^1 f((1-t)u + tv) \, dt \\ &= \alpha F_f(x,y) + \beta F_f(u,v), \end{aligned}$$

which proves the joint convexity of the function F_f.

By Jensen's inequality for the convex function F_f we have

$$\sum_{i=1}^{n} p_i F_f(x_i, y_i) \geq F_f\left(\sum_{i=1}^{n} p_i (x_i, y_i) \right) = F_f\left(\sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i y_i \right),$$

which is equivalent to the first inequality in (8).

By Hermite-Hadamard inequality (HH) we have

$$\begin{aligned} \int_0^1 f\left((1-t) \sum_{i=1}^{n} p_i x_i + t \sum_{i=1}^{n} p_i y_i \right) dt &\geq f\left(\frac{\sum_{i=1}^{n} p_i x_i + \sum_{i=1}^{n} p_i y_i}{2} \right) dt \\ &= f\left(\sum_{i=1}^{n} p_i \left(\frac{x_i + y_i}{2} \right) \right) \end{aligned}$$

and the second part of (8) is proved.

From (HH) we also have for each $i \in \{1, ..., n\}$ that

$$\frac{f(x_i) + f(y_i)}{2} \geq \int_0^1 f[(1-t)x_i + ty_i] \, dt \geq f\left(\frac{x_i + y_i}{2} \right).$$

If we multiply this inequality by $p_i \geq 0$ and sum over i from 1 to n we get the first and second inequality in (9).

The last part in (9) follows by Jensen's inequality.

Let $u := \sum_{i=1}^{n} p_i x_i$ and $v := \sum_{i=1}^{n} p_i y_i$. By Hermite-Hadamard inequality (HH) we also have

$$\frac{f(u) + f(v)}{2} \geq \int_0^1 f[(1-t)u + tv] \, dt,$$

which produces the second inequality in (10).

By Jensen's inequality for f we have

$$\sum_{i=1}^{n} p_i f(x_i) \geq f\left(\sum_{i=1}^{n} p_i x_i\right)$$

and

$$\sum_{i=1}^{n} p_i f(y_i) \geq f\left(\sum_{i=1}^{n} p_i y_i\right).$$

If we sum these two inequalities and divide by 2 we get the first inequality in (10). □

The following result also holds:

Theorem 4. *With the assumptions of Theorem 3 we have*

$$\begin{aligned} 0 &\leq \frac{1}{8}\left[\sum_{i=1}^{n} p_i \left(\nabla_+ f\left(\frac{x_i+y_i}{2}\right)(y_i - x_i)\right)\right. \\ &\left.- \sum_{i=1}^{n} p_i \left(\nabla_- f\left(\frac{x_i+y_i}{2}\right)(y_i - x_i)\right)\right] \\ &\leq \sum_{i=1}^{n} p_i \int_0^1 f\left[(1-t)x_i + ty_i\right] dt - \sum_{i=1}^{n} p_i f\left(\frac{x_i+y_i}{2}\right) \\ &\leq \frac{1}{8}\left[\sum_{i=1}^{n} p_i \left(\nabla_- f(y_i)\right)(y_i - x_i) - \sum_{i=1}^{n} p_i \left(\nabla_+ f(x_i)\right)(y_i - x_i)\right], \end{aligned} \tag{11}$$

and

$$\begin{aligned} 0 &\leq \frac{1}{8}\left[\sum_{i=1}^{n} p_i \left(\nabla_+ f\left(\frac{x_i+y_i}{2}\right)(y_i - x_i)\right)\right. \\ &\left.- \sum_{i=1}^{n} p_i \left(\nabla_- f\left(\frac{x_i+y_i}{2}\right)(y_i - x_i)\right)\right] \\ &\leq \sum_{i=1}^{n} p_i \left(\frac{f(x_i)+f(y_i)}{2}\right) - \sum_{i=1}^{n} p_i \int_0^1 f\left((1-t)x_i + ty_i\right) dt \\ &\leq \frac{1}{8}\left[\sum_{i=1}^{n} p_i \left(\nabla_- f(y_i)\right)(y_i - x_i) - \sum_{i=1}^{n} p_i \left(\nabla_+ f(x_i)\right)(y_i - x_i)\right]. \end{aligned} \tag{12}$$

We also have

$$\begin{aligned} 0 &\leq \frac{1}{8}\left[\nabla_+ f\left(\sum_{i=1}^{n} p_i \left(\frac{x_i+y_i}{2}\right)\right)\left(\sum_{i=1}^{n} p_i (y_i - x_i)\right)\right. \\ &\left.- \nabla_- f\left(\sum_{i=1}^{n} p_i \left(\frac{x_i+y_i}{2}\right)\right)\left(\sum_{i=1}^{n} p_i (y_i - x_i)\right)\right] \\ &\leq \int_0^1 f\left((1-t)\sum_{i=1}^{n} p_i x_i + t\sum_{i=1}^{n} p_i y_i\right) dt - f\left(\sum_{i=1}^{n} p_i \left(\frac{x_i+y_i}{2}\right)\right) \\ &\leq \frac{1}{8}\left[\left(\nabla_- f\left(\sum_{i=1}^{n} p_i y_i\right)\right)\left(\sum_{i=1}^{n} p_i (y_i - x_i)\right)\right. \end{aligned} \tag{13}$$

$$-\left(\nabla_{+}f\left(\sum_{i=1}^{n}p_{i}x_{i}\right)\right)\left(\sum_{i=1}^{n}p_{i}\left(y_{i}-x_{i}\right)\right)\Bigg]$$

and

$$\begin{aligned}0 &\leq \frac{1}{8}\left[\nabla_{+}f\left(\sum_{i=1}^{n}p_{i}\left(\frac{x_{i}+y_{i}}{2}\right)\right)\left(\sum_{i=1}^{n}p_{i}\left(y_{i}-x_{i}\right)\right)\right. \\ &\quad \left.-\nabla_{-}f\left(\sum_{i=1}^{n}p_{i}\left(\frac{x_{i}+y_{i}}{2}\right)\right)\left(\sum_{i=1}^{n}p_{i}\left(y_{i}-x_{i}\right)\right)\right] \\ &\leq \frac{1}{2}\left[f\left(\sum_{i=1}^{n}p_{i}x_{i}\right)+f\left(\sum_{i=1}^{n}p_{i}y_{i}\right)\right] \\ &\quad -\int_{0}^{1}f\left((1-t)\sum_{i=1}^{n}p_{i}x_{i}+t\sum_{i=1}^{n}p_{i}y_{i}\right)dt \\ &\leq \frac{1}{8}\left[\left(\nabla_{-}f\left(\sum_{i=1}^{n}p_{i}y_{i}\right)\right)\left(\sum_{i=1}^{n}p_{i}\left(y_{i}-x_{i}\right)\right)\right. \\ &\quad \left.-\left(\nabla_{+}f\left(\sum_{i=1}^{n}p_{i}x_{i}\right)\right)\left(\sum_{i=1}^{n}p_{i}\left(y_{i}-x_{i}\right)\right)\right].\end{aligned} \tag{14}$$

Proof. From the inequality (3) we have for $a=x_i$ and $b=y_i$, where $i\in\{1,...,n\}$ that

$$\begin{aligned}0 &\leq \frac{1}{8}\left[\nabla_{+}f\left(\frac{x_{i}+y_{i}}{2}\right)\left(y_{i}-x_{i}\right)-\nabla_{-}f\left(\frac{x_{i}+y_{i}}{2}\right)\left(y_{i}-x_{i}\right)\right] \\ &\leq \int_{0}^{1}f\left[(1-t)x_{i}+ty_{i}\right]dt-f\left(\frac{x_{i}+y_{i}}{2}\right) \\ &\leq \frac{1}{8}\left[\left(\nabla_{-}f\left(y_{i}\right)\right)\left(y_{i}-x_{i}\right)-\left(\nabla_{+}f\left(x_{i}\right)\right)\left(y_{i}-x_{i}\right)\right],\end{aligned}$$

for any $i\in\{1,...,n\}$.

If we multiply this inequality by $p_i\geq 0$ and sum over i from 1 to n, then we get

$$\begin{aligned}0 &\leq \frac{1}{8}\sum_{i=1}^{n}p_{i}\left[\nabla_{+}f\left(\frac{x_{i}+y_{i}}{2}\right)\left(y_{i}-x_{i}\right)-\nabla_{-}f\left(\frac{x_{i}+y_{i}}{2}\right)\left(y_{i}-x_{i}\right)\right] \\ &\leq \sum_{i=1}^{n}p_{i}\int_{0}^{1}f\left[(1-t)x_{i}+ty_{i}\right]dt-\sum_{i=1}^{n}p_{i}f\left(\frac{x_{i}+y_{i}}{2}\right) \\ &\leq \frac{1}{8}\sum_{i=1}^{n}p_{i}\left[\left(\nabla_{-}f\left(y_{i}\right)\right)\left(y_{i}-x_{i}\right)-\left(\nabla_{+}f\left(x_{i}\right)\right)\left(y_{i}-x_{i}\right)\right],\end{aligned}$$

which is equivalent to (11).

The inequality (12) follows in a similar way by employing the inequality (6).

The inequalities (13) and (14) follow by taking $a=\sum_{i=1}^{n}p_{i}x_{i}$ and $b=\sum_{i=1}^{n}p_{i}y_{i}$ in the inequalities (3) and (6). □

3. Examples for Norms

Now, assume that $(X,\|\cdot\|)$ is a normed linear space. The function $f_0(x)=\frac{1}{2}\|x\|^2$, $x\in X$ is convex and thus the following limits exist

(iv) $\langle x, y\rangle_s := (\nabla_+ f_0(y))(x) = \lim_{t\to 0+}\left[\frac{\|y+tx\|^2 - \|y\|^2}{2t}\right]$;

(v) $\langle x, y\rangle_i := (\nabla_- f_0(y))(x) = \lim_{s\to 0-}\left[\frac{\|y+sx\|^2 - \|y\|^2}{2s}\right]$;

for any x, $y \in X$. They are called the *lower* and *upper semi-inner* products associated to the norm $\|\cdot\|$.

For the sake of completeness we list here some of the main properties of these mappings that will be used in the sequel (see for example [7] or [8]), assuming that p, $q \in \{s, i\}$ and $p \neq q$:

(a) $\langle x, x\rangle_p = \|x\|^2$ for all $x \in X$;
(aa) $\langle \alpha x, \beta y\rangle_p = \alpha\beta \langle x, y\rangle_p$ if α, $\beta \geq 0$ and x, $y \in X$;
(aaa) $\left|\langle x, y\rangle_p\right| \leq \|x\|\,\|y\|$ for all x, $y \in X$;
(av) $\langle \alpha x + y, x\rangle_p = \alpha \langle x, x\rangle_p + \langle y, x\rangle_p$ if x, $y \in X$ and $\alpha \in \mathbb{R}$;
(v) $\langle -x, y\rangle_p = -\langle x, y\rangle_q$ for all x, $y \in X$;
(va) $\langle x + y, z\rangle_p \leq \|x\|\,\|z\| + \langle y, z\rangle_p$ for all x, y, $z \in X$;
(vaa) The mapping $\langle \cdot, \cdot\rangle_p$ is continuous and subadditive (superadditive) in the first variable for $p = s$ (or $p = i$);
(vaaa) The normed linear space $(X, \|\cdot\|)$ is smooth at the point $x_0 \in X\setminus\{0\}$ if and only if $\langle y, x_0\rangle_s = \langle y, x_0\rangle_i$ for all $y \in X$; in general $\langle y, x\rangle_i \leq \langle y, x\rangle_s$ for all x, $y \in X$;
(ax) If the norm $\|\cdot\|$ is induced by an inner product $\langle \cdot, \cdot\rangle$, then $\langle y, x\rangle_i = \langle y, x\rangle = \langle y, x\rangle_s$ for all x, $y \in X$.

Applying inequality (HH) for the convex function $f_r(x) = \|x\|^r$, $r \geq 1$ one may deduce the inequality

$$\left\|\frac{x+y}{2}\right\|^r \leq \int_0^1 \|(1-t)x + ty\|^r\,dt \leq \frac{\|x\|^r + \|y\|^r}{2} \tag{15}$$

for any $x, y \in X$.

Let $(X, \|\cdot\|)$ be a normed linear space and $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n)$ be n-tuples of vectors in X, then for the probability distribution $p = (p_1, ..., p_n)$ and $r \geq 1$ we have by Theorem 3 for the convex function $f(x) = \|x\|^r$ that

$$\begin{aligned}\sum_{i=1}^n p_i \int_0^1 \|(1-t)x_i + ty_i\|^r\,dt &\geq \int_0^1 \left\|(1-t)\sum_{i=1}^n p_i x_i + t\sum_{i=1}^n p_i y_i\right\|^r dt \\ &\geq \left\|\sum_{i=1}^n p_i\left(\frac{x_i+y_i}{2}\right)\right\|^r,\end{aligned} \tag{16}$$

$$\begin{aligned}\sum_{i=1}^n p_i\left(\frac{\|x_i\|^r + \|y_i\|^r}{2}\right) &\geq \sum_{i=1}^n \int_0^1 p_i \|(1-t)x_i + ty_i\|^r\,dt \\ &\geq \sum_{i=1}^n p_i \left\|\frac{x_i+y_i}{2}\right\|^r \geq \left\|\sum_{i=1}^n p_i\left(\frac{x_i+y_i}{2}\right)\right\|^r\end{aligned} \tag{17}$$

and

$$\begin{aligned}\sum_{i=1}^n p_i\left(\frac{\|x_i\|^r + \|y_i\|^r}{2}\right) &\geq \frac{1}{2}\left[\left\|\sum_{i=1}^n p_i x_i\right\|^r + \left\|\sum_{i=1}^n p_i y_i\right\|^r\right] \\ &\geq \int_0^1 \left\|(1-t)\sum_{i=1}^n p_i x_i + t\sum_{i=1}^n p_i y_i\right\|^r dt.\end{aligned} \tag{18}$$

If we use Theorem 4 for the convex function $f(x) = \frac{1}{2}\|x\|^2$ then for $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n)$ n-tuples of vectors in X and for the probability distribution $p = (p_1, ..., p_n)$ we have

$$0 \leq \frac{1}{4}\left[\sum_{k=1}^n p_k \left\langle y_k - x_k, \frac{x_k+y_k}{2}\right\rangle_s - \sum_{k=1}^n p_k \left\langle y_k - x_k, \frac{x_k+y_k}{2}\right\rangle_i\right] \tag{19}$$

$$\leq \sum_{k=1}^{n} p_k \int_0^1 \|(1-t)\, x_k + t y_k\|^2\, dt - \sum_{k=1}^{n} p_k \left\|\frac{x_k + y_k}{2}\right\|^2$$
$$\leq \frac{1}{4}\left[\sum_{k=1}^{n} p_k \langle y_k - x_k, y_k\rangle_i - \sum_{k=1}^{n} p_k \langle y_k - x_k, x_k\rangle_s\right],$$

and

$$0 \leq \frac{1}{4}\left[\sum_{k=1}^{n} p_k \left\langle y_k - x_k, \frac{x_k + y_k}{2}\right\rangle_s - \sum_{k=1}^{n} p_k \left\langle y_k - x_k, \frac{x_k + y_k}{2}\right\rangle_i\right] \tag{20}$$
$$\leq \sum_{k=1}^{n} p_k \left(\frac{\|x_k\|^2 + \|y_k\|^2}{2}\right) - \sum_{k=1}^{n} p_k \int_0^1 \|(1-t)\, x_k + t y_k\|^2\, dt$$
$$\leq \frac{1}{4}\left[\sum_{k=1}^{n} p_k \langle y_k - x_k, y_k\rangle_i - \sum_{k=1}^{n} p_k \langle y_k - x_k, x_k\rangle_s\right].$$

We also have

$$0 \leq \frac{1}{4}\left[\left\langle \sum_{k=1}^{n} p_k (y_k - x_k), \sum_{k=1}^{n} p_k \left(\frac{x_k + y_k}{2}\right)\right\rangle_s \right. \tag{21}$$
$$\left. - \left\langle \sum_{k=1}^{n} p_k (y_k - x_k), \sum_{k=1}^{n} p_k \left(\frac{x_k + y_k}{2}\right)\right\rangle_i\right]$$
$$\leq \int_0^1 \left\|(1-t)\sum_{k=1}^{n} p_k x_k + t \sum_{k=1}^{n} p_k y_k\right\|^2 dt - \left\|\sum_{k=1}^{n} p_k \left(\frac{x_k + y_k}{2}\right)\right\|^2$$
$$\leq \frac{1}{4}\left[\left\langle \sum_{k=1}^{n} p_k (y_k - x_k), \sum_{k=1}^{n} p_k y_k\right\rangle_i - \left\langle \sum_{k=1}^{n} p_k (y_k - x_k), \sum_{k=1}^{n} p_k x_k\right\rangle_s\right]$$

and

$$0 \leq \frac{1}{4}\left[\left\langle \sum_{k=1}^{n} p_k (y_k - x_k), \sum_{k=1}^{n} p_k \left(\frac{x_k + y_k}{2}\right)\right\rangle_s \right. \tag{22}$$
$$\left. - \left\langle \sum_{k=1}^{n} p_k (y_k - x_k), \sum_{k=1}^{n} p_k \left(\frac{x_k + y_k}{2}\right)\right\rangle_i\right]$$
$$\leq \frac{1}{2}\left[\left\|\sum_{k=1}^{n} p_k x_k\right\|^2 + \left\|\sum_{k=1}^{n} p_k y_k\right\|^2\right]$$
$$- \int_0^1 \left\|(1-t)\sum_{k=1}^{n} p_k x_k + t \sum_{k=1}^{n} p_k y_k\right\|^2 dt$$
$$\leq \frac{1}{4}\left[\left\langle \sum_{k=1}^{n} p_k (y_k - x_k), \sum_{k=1}^{n} p_k y_k\right\rangle_i - \left\langle \sum_{k=1}^{n} p_k (y_k - x_k), \sum_{k=1}^{n} p_k x_k\right\rangle_s\right].$$

4. Examples for Functions of a Real Variable

If $f : I \to \mathbb{R}$ is convex on the interval I and $p_i \geq 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} p_i = 1$, then

$$\sum_{i=1}^{n} p_i \int_0^1 f\left((1-t)x_i + ty_i\right) dt \geq \int_0^1 f\left((1-t)\sum_{i=1}^{n} p_i x_i + t\sum_{i=1}^{n} p_i y_i\right) dt \tag{23}$$

$$\sum_{i=1}^{n} p_i \left(\frac{f(x_i) + f(y_i)}{2}\right) \geq \sum_{i=1}^{n} p_i \int_0^1 f\left((1-t)x_i + ty_i\right) dt. \tag{24}$$

If $f : I \to \mathbb{R}$ is convex and differentiable on the interior of $\mathring{I}$ then for all $x_i \in \mathring{I}$ and $p_i \geq 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} p_i = 1$, then by (11) and (12) we get

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n} p_i \int_0^1 f\left[(1-t)x_i + ty_i\right] dt - \sum_{i=1}^{n} p_i f\left(\frac{x_i + y_i}{2}\right) \\ &\leq \frac{1}{8}\sum_{i=1}^{n} p_i \left[f'(y_i) - f'(x_i)\right](y_i - x_i), \end{aligned} \tag{25}$$

and

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n} p_i \left(\frac{f(x_i) + f(y_i)}{2}\right) - \sum_{i=1}^{n} p_i \int_0^1 f\left((1-t)x_i + ty_i\right) dt \\ &\leq \frac{1}{8}\sum_{i=1}^{n} p_i \left[f'(y_i) - f'(x_i)\right](y_i - x_i). \end{aligned} \tag{26}$$

If $f(t) = \frac{1}{t}$ with $t > 0$, then for $y_i \neq x_i$, $i \in \{1, ..., n\}$ we have

$$\int_0^1 f\left((1-t)x_i + ty_i\right) dt = \int_0^1 \frac{1}{(1-t)x_i + ty_i} dt = \frac{\ln y_i - \ln x_i}{y_i - x_i}$$

and

$$\int_0^1 \left((1-t)\sum_{i=1}^{n} p_i x_i + t\sum_{i=1}^{n} p_i y_i\right)^{-1} dt = \frac{\ln\left(\sum_{i=1}^{n} p_i x_i\right) - \ln\left(\sum_{i=1}^{n} p_i y_i\right)}{\sum_{i=1}^{n} p_i x_i - \sum_{i=1}^{n} p_i y_i},$$

provided $\sum_{i=1}^{n} p_i x_i \neq \sum_{i=1}^{n} p_i y_i$.

From (23) we get

$$\sum_{i=1}^{n} p_i \frac{\ln y_i - \ln x_i}{y_i - x_i} \geq \frac{\ln\left(\sum_{i=1}^{n} p_i x_i\right) - \ln\left(\sum_{i=1}^{n} p_i y_i\right)}{\sum_{i=1}^{n} p_i x_i - \sum_{i=1}^{n} p_i y_i}$$

that is equivalent to

$$\ln\left(\prod_{i=1}^{n}\left(\frac{y_i}{x_i}\right)^{\frac{p_i}{y_i - x_i}}\right) \geq \ln\left[\left(\frac{\sum_{i=1}^{n} p_i x_i}{\sum_{i=1}^{n} p_i y_i}\right)^{\frac{1}{\sum_{i=1}^{n} p_i x_i - \sum_{i=1}^{n} p_i y_i}}\right]$$

and to

$$\prod_{i=1}^{n}\left(\frac{y_i}{x_i}\right)^{\frac{p_i}{y_i - x_i}} \geq \left(\frac{\sum_{i=1}^{n} p_i x_i}{\sum_{i=1}^{n} p_i y_i}\right)^{\frac{1}{\sum_{i=1}^{n} p_i x_i - \sum_{i=1}^{n} p_i y_i}}. \tag{27}$$

From (24) we get in a similar way that

$$\exp\left[\sum_{i=1}^{n} p_i \left(\frac{x_i + y_i}{2x_i y_i}\right)\right] \geq \prod_{i=1}^{n}\left(\frac{y_i}{x_i}\right)^{\frac{p_i}{y_i - x_i}}, \tag{28}$$

from (25) we get

$$1 \le \frac{\prod_{i=1}^{n}\left(\frac{y_i}{x_i}\right)^{\frac{p_i}{y_i-x_i}}}{\exp\left[\sum_{i=1}^{n} p_i\left(\frac{2}{x_i+y_i}\right)\right]} \le \exp\left(\frac{1}{8}\sum_{i=1}^{n} p_i \frac{(x_i+y_i)(y_i-x_i)^2}{x_i^2 y_i^2}\right) \tag{29}$$

and from (26) we get

$$1 \le \frac{\exp\left[\sum_{i=1}^{n} p_i\left(\frac{x_i+y_i}{2x_i y_i}\right)\right]}{\prod_{i=1}^{n}\left(\frac{y_i}{x_i}\right)^{\frac{p_i}{y_i-x_i}}} \le \exp\left(\frac{1}{8}\sum_{i=1}^{n} p_i \frac{(x_i+y_i)(y_i-x_i)^2}{x_i^2 y_i^2}\right). \tag{30}$$

The interested reader may apply some of the above inequalities for other instances of convex functions such as $f(t) = -\ln t$, $t\ln t$, $\exp t$ etc... and we omit the details.

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