Advances in Analysis and Applied Mathematics, **2**(1) (2025), 1–20.

https://doi.org/10.62298/advmath.22
ISSN Online: 3062-0686 / Open Access

Research Paper

A Collection of Trigonometric Inequalities Using Integral Methods

Christophe Chesneau[1,*]

[1]Department of Mathematics, LMNO, University of Caen-Normandie, 14032, Caen, France
[*]To whom correspondence should be addressed: christophe.chesneau@gmail.com

Received 30 April 2025; Revised 12 June 2025; Accepted 26 June 2025; Published 30 June 2025

Abstract

This article explores the use of integral methods in the proof of trigonometric inequalities. Although classical methods such as induction, convexity, and series expansions are well established, the manipulation of integrals to establish sharp trigonometric inequalities remains clearly underexploited. In particular, we use primitive techniques, the Chebyshev integral inequality, and the Jensen integral inequality to recover known results, including the Jordan, Kober, and Cusa-Huygens inequalities. New inequalities are also derived and discussed. The corresponding proofs are given in detail for the sake of completeness. Some figures illustrate selected two-sided inequalities. In final, this article provides a collection of trigonometric inequalities suitable for advanced teaching and further research in many areas of analysis.

Key Words: Trigonometric inequalities, Jordan inequality, Kober inequality, Cusa-Huygens inequality, Chebyshev integral inequality, Jensen integral inequality.
AMS 2020 Classification: 26D15

1. Introduction

1.1. Context

Trigonometric inequalities play an important role in mathematical analysis, approximation theory and various branches of applied mathematics. Among the classical results in this area is the Jordan inequality, which gives a sharp lower bound for the sine function, i.e., for any $x \in (0, \pi/2)$,

$$\frac{2}{\pi}x < \sin(x).$$

This result is particularly useful in approximation problems and in determining the behavior of the sine function near the origin, i.e., $x \in (0, \epsilon)$ with small ϵ. An analogous inequality centered around the cosine function is the Kober inequality which states that, for any $x \in (0, \pi/2)$,

$$1 - \frac{2}{\pi}x < \cos(x).$$

Also remarkable is the Cusa-Huygens inequality, which offers a refinement involving a special weighted mean of the cosine function, i.e., for any $x \in (0, \pi/2)$,

$$\frac{\sin(x)}{x} > \frac{2 + \cos(x)}{3}.$$

Originally derived by Cusa in the fifteenth century and rigorously proved by Huygens two centuries later, this inequality has inspired much research focused on its refinements and generalizations.

Each of these inequalities provides information about the structure and behavior of trigonometric functions. They serve as basic tools for deriving more sophisticated analytic results. Their generalizations often involve the introduction of parameters, extensions to hyperbolic functions, and applications of power means, convexity, and other methods of functional analysis.

We refer the interested reader to the reference book of Mitrinović [1] and the following notable articles: [2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15]. All of them contain results and extensive discussions on the refinements, generalizations and applications of these classical trigonometric inequalities.

1.2. Contributions

Several proof techniques have been developed to establish trigonometric inequalities. These include mathematical induction, classical calculus tools such as monotonicity and convexity analysis, power series expansions, differential inequalities, the theory of means, and more advanced tools from real and functional analysis. Each of these approaches has played a significant role in the development and refinement of these inequalities.

Despite this diversity of methods, the use of integrals in proving trigonometric inequalities remains relatively limited in the literature. This article aims to address this gap. We show how elementary integral tools, particularly the fundamental theorem of calculus (primitive techniques), Chebyshev integral inequality, and Jensen integral inequality, can be used effectively to recover known results such as the Jordan, Kober, and Cusa-Huygens inequalities. More importantly, these tools can also be employed to derive new and original trigonometric inequalities.

All proofs are presented in full detail. They are accessible for teaching purposes and can serve as a basis for further research. Where appropriate, some figures are displayed, showing the sharpness of selected inequalities.

The integral-based method developed here offers a promising framework for extending the scope of trigonometric inequalities to new mathematical horizons.

1.3. Article organization

The article is divided into six main sections, followed by an appendix. Each section presents a set of trigonometric inequalities, arranged in increasing order of technical complexity. Section 2 introduces the first results based on simple integral methods. Section 3 is devoted to trigonometric inequalities derived from the Chebyshev integral inequality. Section 4 combines integral methods with the concavity properties of the sine and cosine functions. Section 5 focuses on inequalities derived from the Jensen integral inequality. Section 6 examines special inequalities involving double integrals. Section 7 contains concluding remarks. The appendix presents the general forms of the Chebyshev integral inequality and the Jensen integral inequality, which are central tools in our proofs.

2. First Set of Trigonometric Inequalities

2.1. Statement

Some well-known inequalities, including the Jordan and Kober inequalities, are examined in the proposition below. The main interest remains the proof using simple integral methods.

Proposition 1. *The integral method can be used to prove the classical trigonometric inequalities below.*

1. *For any $x \in (0, \pi/2)$, we have*

$$\sin(x) < x.$$

2. *For any* $x \in (0, \pi/2)$*, we have*
$$\sin(x) > x\cos(x).$$
3. *For any* $x \in (0, \pi/2)$*, we have*
$$\sin(x) > \frac{2}{\pi}x.$$
4. *For any* $x \in (0, \pi/2)$*, we have*
$$\cos(x) > 1 - \frac{2}{\pi}x.$$

Proof of Proposition 1. The proof is based on basic integral representations of the sine and cosine functions, known inequalities and monotonicity properties.

1. A simple integral development gives
$$\sin(x) = x\int_0^1 \cos(ux)du.$$
For any $u \in (0, 1)$ and $x \in (0, \pi/2)$, we have $\cos(ux) < 1$. We immediately derive
$$\sin(x) = x\int_0^1 \cos(ux)du < x\int_0^1 du = x.$$
2. We use again
$$\sin(x) = x\int_0^1 \cos(ux)du.$$
For any $u \in (0, 1)$ and $x \in (0, \pi/2)$, $\cos(ux)$ is strictly decreasing with respect to u. As a result, we have
$$\sin(x) = x\int_0^1 \cos(ux)du > x\cos(1 \times x)\int_0^1 du = x\cos(x).$$
3. We use again
$$\sin(x) = x\int_0^1 \cos(ux)du.$$
For any $u \in (0, 1)$ and $x \in (0, \pi/2)$, $\cos(ux)$ is strictly decreasing with respect to x. Using this and an integral calculus, i.e., $\int_0^1 \cos(u(\pi/2))\, du = 2/\pi$, we obtain
$$\sin(x) = x\int_0^1 \cos(ux)du > x\int_0^1 \cos\left(u \times \frac{\pi}{2}\right) du = \frac{2}{\pi}x.$$
4. A basic integral calculus gives
$$\cos(x) = 1 - x\int_0^1 \sin(ux)du. \tag{1}$$
For any $u \in (0, 1)$ and $x \in (0, \pi/2)$, $\sin(ux)$ is strictly increasing with respect to u. This and an integral calculus yields
$$\cos(x) = 1 - x\int_0^1 \sin(ux)du > 1 - x\int_0^1 \sin\left(u \times \frac{\pi}{2}\right) du = 1 - \frac{2}{\pi}x.$$

This ends the proof of Proposition 1. □

The third item of this proposition is the Jordan inequality, and the fourth presents the Kober inequality. The proofs using simple integral methods give a clear mathematical understanding of these famous results, with the same mathematical foundation.

The proposition below considers a more general one-parameter integral method. Its main interest is that the integral can also be expressed in terms of polynomial-trigonometric functions, as will be shown later. The proof is mainly based on the following basic trigonometric inequality: $\sin(x) < 1$ for any $x \in (0, \pi/2)$ and Proposition 1.

Proposition 2. *The integral method can be used to prove the trigonometric inequalities below.*

1. *For any $\alpha \geq 0$ and $x \in (0, \pi/2)$, we have*

$$\int_0^1 u^\alpha \sin(ux)du < \frac{1}{\alpha+1} \min\left(1, \frac{\alpha+1}{\alpha+2}x\right).$$

2. *For any $\alpha \geq 0$ and $x \in (0, \pi/2)$, we have*

$$\int_0^1 u^\alpha \sin(ux)du > \frac{1}{\alpha+2} x\cos(x).$$

3. *For any $\alpha \geq 0$ and $x \in (0, \pi/2)$, we have*

$$\int_0^1 u^\alpha \sin(ux)du > \frac{1}{\alpha+1} \times \frac{2}{\pi}.$$

4. *For any $\alpha \geq 0$ and $x \in (0, \pi/2)$, we have*

$$\int_0^1 u^\alpha \cos(ux)du > \frac{1}{\alpha+1} \left(1 - \frac{2}{\pi} \times \frac{\alpha+1}{\alpha+2}x\right).$$

Proof of Proposition 2.

1. For any $u \in (0, 1)$, $\alpha \geq 0$ and $x \in (0, \pi/2)$, we have $\sin(ux) < 1$. So, we derive

$$\int_0^1 u^\alpha \sin(ux)du < \int_0^1 u^\alpha du = \frac{1}{\alpha+1}.$$

   On the other hand, for any $u \in (0, 1)$, $\alpha \geq 0$ and $x \in (0, \pi/2)$, the first item of Proposition 1 gives $\sin(ux) < ux$, which implies that

$$\int_0^1 u^\alpha \sin(ux)du < \int_0^1 u^\alpha \times uxdu = x \int_0^1 u^{\alpha+1} du = \frac{1}{\alpha+2}x.$$

   As a result, we have

$$\int_0^1 u^\alpha \sin(ux)du < \min\left(\frac{1}{\alpha+1}, \frac{1}{\alpha+2}x\right) = \frac{1}{\alpha+1} \min\left(1, \frac{\alpha+1}{\alpha+2}x\right).$$

2. For any $u \in (0, 1)$, $\alpha \geq 0$ and $x \in (0, \pi/2)$, it follows from the second item of Proposition 1 that $\sin(ux) > ux\cos(ux)$. This and the fact that $\cos(ux)$ is strictly decreasing with respect to u give

$$\int_0^1 u^\alpha \sin(ux)du > \int_0^1 u^\alpha \times ux\cos(ux)du = x \int_0^1 u^{\alpha+1} \cos(ux)du > x\cos(x) \int_0^1 u^{\alpha+1} du = \frac{1}{\alpha+2} x\cos(x).$$

3. For any $u \in (0, 1)$, $\alpha \geq 0$ and $x \in (0, \pi/2)$, the third item of Proposition 1 ensures that $\sin(ux) > (2/\pi)ux$. So, we have

$$\int_0^1 u^\alpha \sin(ux)du > \int_0^1 u^\alpha \times \frac{2}{\pi}uxdu = \frac{2}{\pi}x \int_0^1 u^{\alpha+1} du = \frac{1}{\alpha+2} \times \frac{2}{\pi}x.$$

4. For any $u \in (0,1)$, $\alpha \geq 0$ and $x \in (0, \pi/2)$, the fourth item of Proposition 1 gives $1 - \cos(ux) > 1 - (2/\pi)ux$. So, we have

$$\int_0^1 u^\alpha \cos(ux)du > \int_0^1 u^\alpha \times \left(1 - \frac{2}{\pi}ux\right) du = \int_0^1 u^\alpha du - \frac{2}{\pi}x \int_0^1 u^{\alpha+1} du$$
$$= \frac{1}{\alpha+1} - \frac{2}{\pi} \times \frac{1}{\alpha+2}x = \frac{1}{\alpha+1}\left(1 - \frac{2}{\pi} \times \frac{\alpha+1}{\alpha+2}x\right).$$

This concludes the proof of Proposition 2. □

Note that, based on the first and two items, for any $\alpha \geq 0$ and $x \in (0, \pi/2)$, the following two-sided inequality holds:

$$\frac{1}{\alpha+2}x\cos(x) < \int_0^1 u^\alpha \sin(ux)du < \frac{1}{\alpha+1}\min\left(1, \frac{\alpha+1}{\alpha+2}x\right).$$

Tables 1 and 2 present some special cases of interest of this proposition, focusing on the first integer values of α; Table 1 concerns the first and second items, while Table 2 concerns the third and fourth items.

α	$\int_0^1 u^\alpha \sin(ux)du$	$< \frac{1}{\alpha+1}\min\left(1, \frac{\alpha+1}{\alpha+2}x\right)$
0	$\frac{1-\cos(x)}{x}$	$< \min\left(1, \frac{1}{2}x\right)$
1	$\frac{\sin(x) - x\cos(x)}{x^2}$	$< \frac{1}{2}\min\left(1, \frac{2}{3}x\right)$
2	$\frac{(2-x^2)\cos(x) + 2x\sin(x) - 2}{x^3}$	$< \frac{1}{3}\min\left(1, \frac{3}{4}x\right)$
3	$\frac{3(x^2-2)\sin(x) - x(x^2-6)\cos(x)}{x^4}$	$< \frac{1}{4}\min\left(1, \frac{4}{5}x\right)$
4	$\frac{4x(x^2-6)\sin(x) - (x^4 - 12x^2 + 24)\cos(x) + 24}{x^5}$	$< \frac{1}{5}\min\left(1, \frac{5}{6}x\right)$

α	$\int_0^1 u^\alpha \sin(ux)du$	$> \frac{1}{\alpha+2}x\cos(x)$
0	$\frac{1-\cos(x)}{x}$	$> \frac{1}{2}x\cos(x)$
1	$\frac{\sin(x) - x\cos(x)}{x^2}$	$> \frac{1}{3}x\cos(x)$
2	$\frac{(2-x^2)\cos(x) + 2x\sin(x) - 2}{x^3}$	$> \frac{1}{4}x\cos(x)$
3	$\frac{3(x^2-2)\sin(x) - x(x^2-6)\cos(x)}{x^4}$	$> \frac{1}{5}x\cos(x)$
4	$\frac{4x(x^2-6)\sin(x) - (x^4 - 12x^2 + 24)\cos(x) + 24}{x^5}$	$> \frac{1}{6}x\cos(x)$

Table 1. Some special cases of interest of the first and second items of Proposition 2

α	$\int_0^1 u^\alpha \sin(ux)du$	$> \frac{1}{\alpha+2} \times \frac{2}{\pi}x$
0	$\frac{1-\cos(x)}{x}$	$> \frac{1}{2} \times \frac{2}{\pi}x$
1	$\frac{\sin(x) - x\cos(x)}{x^2}$	$> \frac{1}{3} \times \frac{2}{\pi}x$
2	$\frac{(2-x^2)\cos(x) + 2x\sin(x) - 2}{x^3}$	$> \frac{1}{4} \times \frac{2}{\pi}x$
3	$\frac{3(x^2-2)\sin(x) - x(x^2-6)\cos(x)}{x^4}$	$> \frac{1}{5} \times \frac{2}{\pi}x$
4	$\frac{4x(x^2-6)\sin(x) - (x^4-12x^2+24)\cos(x) + 24}{x^5}$	$> \frac{1}{6} \times \frac{2}{\pi}x$
α	$\int_0^1 u^\alpha \cos(ux)du$	$> \frac{1}{\alpha+1}\left(1 - \frac{2}{\pi} \times \frac{\alpha+1}{\alpha+2}x\right)$
0	$\frac{\sin(x)}{x}$	$> 1 - \frac{2}{\pi} \times \frac{1}{2}x$
1	$\frac{x\sin(x) + \cos(x) - 1}{x^2}$	$> \frac{1}{2}\left(1 - \frac{2}{\pi} \times \frac{2}{3}x\right)$
2	$\frac{(x^2-2)\sin(x) + 2x\cos(x)}{x^3}$	$> \frac{1}{3}\left(1 - \frac{2}{\pi} \times \frac{3}{4}x\right)$
3	$\frac{x(x^2-6)\sin(x) + 3(x^2-2)\cos(x) + 6}{x^4}$	$> \frac{1}{4}\left(1 - \frac{2}{\pi} \times \frac{4}{5}x\right)$
4	$\frac{4x(x^2-6)\cos(x) + (x^4-12x^2+24)\sin(x)}{x^5}$	$> \frac{1}{5}\left(1 - \frac{2}{\pi} \times \frac{5}{6}x\right)$

Table 2. Some special cases of interest of the third and fourth items of Proposition 2

To the best of our knowledge, the inequalities obtained for $\alpha = 2, 3, 4$, in addition to those obtained for the decimal values, are new in the literature.

2.2. Focus on the cases $\alpha = 0$ and $\alpha = 1$

Applying the third item of Proposition 2 to $\alpha = 0$, for any $x \in (0, \pi/2)$, we obtain

$$\frac{1-\cos(x)}{x} > \frac{1}{2} \times \frac{2}{\pi}x = \frac{1}{\pi}x,$$

from which we derive

$$\cos(x) < 1 - \frac{1}{\pi}x^2.$$

This is the "second-type" Kober inequality.

Applying the fourth item of Proposition 2 to $\alpha = 0$, for any $x \in (0, \pi/2)$, we obtain

$$\frac{\sin(x)}{x} > 1 - \frac{2}{\pi} \times \frac{1}{2}x = 1 - \frac{1}{\pi}x.$$

Applying the two first items in Proposition 2 to $\alpha = 1$, for any $x \in (0, \pi/2)$, we get the following two-sided inequality:

$$\frac{1}{3}x\cos(x) < \frac{\sin(x) - x\cos(x)}{x^2} < \frac{1}{2}\min\left(1, \frac{2}{3}x\right),$$

from which we derive

$$\frac{1}{3}x^3\cos(x) < \sin(x) - x\cos(x) < \frac{1}{2}x^2\min\left(1, \frac{2}{3}x\right). \tag{2}$$

In particular, the left-hand side inequality gives $\sin(x) - x\cos(x) > (1/3)x^3\cos(x) > 0$, which allows us to recover the second item of Proposition 1. This two-sided inequality is of interest because the difference function $\sin(x) - x\cos(x)$ naturally appears in various estimates and approximations. It often arises in the analysis of trigonometric integrals and in the study of convexity properties of related functions.

3. Second Set of Trigonometric Inequalities

3.1. Statement

The statement below presents a new set of one-parameter trigonometric inequalities. The proof relies on the monotonicity of the functions involved and the Chebyshev integral inequality, which is recalled in full generality in Appendix.

Proposition 3. *The integral method can be used to prove the trigonometric inequalities below.*

1. *For any $\alpha > 0$ and $x \in (0, \pi/2)$, we have*
$$\int_0^1 u^\alpha \cos(ux)du < \frac{1}{\alpha+1} \times \frac{\sin(x)}{x}.$$
The equality holds for $\alpha = 0$.
2. *For any $\alpha > 0$ and $x \in (0, \pi/2)$, we have*
$$\int_0^1 u^\alpha \sin(ux)du > \frac{1}{\alpha+1} \times \frac{1-\cos(x)}{x}.$$
The equality holds for $\alpha = 0$.

Proof of Proposition 3.

1. For any $u \in (0,1)$, $\alpha > 0$ and $x \in (0, \pi/2)$, u^α is strictly increasing and $\cos(ux)$ is strictly decreasing with respect to u. They are therefore of opposite monotonicity. The Chebyshev integral inequality applied to these two functions gives
$$\frac{1}{1-0}\int_0^1 u^\alpha \cos(ux)du < \left[\frac{1}{1-0}\int_0^1 u^\alpha du\right]\left[\frac{1}{1-0}\int_0^1 \cos(ux)du\right],$$
so that
$$\int_0^1 u^\alpha \cos(ux)du < \frac{1}{\alpha+1} \times \frac{\sin(x)}{x}.$$
For $\alpha = 0$, we obviously have
$$\int_0^1 u^\alpha \cos(ux)du = \int_0^1 \cos(ux)du = \frac{\sin(x)}{x} = \frac{1}{\alpha+1} \times \frac{\sin(x)}{x}.$$
2. For any $u \in (0,1)$, $\alpha > 0$ and $x \in (0, \pi/2)$, u^α is strictly increasing and $\sin(ux)$ is strictly increasing with respect to u. They are therefore of the same monotonicity. The Chebyshev integral inequality applied to these two functions gives
$$\frac{1}{1-0}\int_0^1 u^\alpha \sin(ux)du > \left[\frac{1}{1-0}\int_0^1 u^\alpha du\right]\left[\frac{1}{1-0}\int_0^1 \sin(ux)du\right],$$
so that
$$\int_0^1 u^\alpha \sin(ux)du > \frac{1}{\alpha+1} \times \frac{1-\cos(x)}{x}.$$

For $\alpha = 0$, we obviously have

$$\int_0^1 u^\alpha \sin(ux)du = \int_0^1 \sin(ux)du = \frac{1-\cos(x)}{x} = \frac{1}{\alpha+1} \times \frac{1-\cos(x)}{x}.$$

This ends the proof of Proposition 3. □

Table 3 presents some special cases of interest of this proposition, focusing on the first integer values of α.

α	$\int_0^1 u^\alpha \cos(ux)du$	$< \frac{1}{\alpha+1} \times \frac{\sin(x)}{x}$
1	$\frac{x\sin(x)+\cos(x)-1}{x^2}$	$< \frac{1}{2} \times \frac{\sin(x)}{x}$
2	$\frac{(x^2-2)\sin(x)+2x\cos(x)}{x^3}$	$< \frac{1}{3} \times \frac{\sin(x)}{x}$
3	$\frac{x(x^2-6)\sin(x)+3(x^2-2)\cos(x)+6}{x^4}$	$< \frac{1}{4} \times \frac{\sin(x)}{x}$
4	$\frac{4x(x^2-6)\cos(x)+(x^4-12x^2+24)\sin(x)}{x^5}$	$< \frac{1}{5} \times \frac{\sin(x)}{x}$
α	$\int_0^1 u^\alpha \sin(ux)du$	$> \frac{1}{\alpha+1} \times \frac{1-\cos(x)}{x}$
1	$\frac{\sin(x)-x\cos(x)}{x^2}$	$> \frac{1}{2} \times \frac{1-\cos(x)}{x}$
2	$\frac{(2-x^2)\cos(x)+2x\sin(x)-2}{x^3}$	$> \frac{1}{3} \times \frac{1-\cos(x)}{x}$
3	$\frac{3(x^2-2)\sin(x)-x(x^2-6)\cos(x)}{x^4}$	$> \frac{1}{4} \times \frac{1-\cos(x)}{x}$
4	$\frac{4x(x^2-6)\sin(x)-(x^4-12x^2+24)\cos(x)+24}{x^5}$	$> \frac{1}{5} \times \frac{1-\cos(x)}{x}$

Table 3. Some special cases of interest of Proposition 3

To the best of our knowledge, the inequalities obtained for $\alpha = 1, 2, 3, 4$, in addition to those obtained for the decimal values, are new in the literature.

3.2. Focus on the case $\alpha = 1$

Applying the first item of Proposition 3 to $\alpha = 1$, for any $x \in (0, \pi/2)$, we get

$$\frac{x\sin(x)+\cos(x)-1}{x^2} < \frac{1}{2} \times \frac{\sin(x)}{x},$$

so that

$$x^2\sin(x) < 2[1-\cos(x)].$$

The second item gives

$$\frac{\sin(x)-x\cos(x)}{x^2} > \frac{1}{2} \times \frac{1-\cos(x)}{x},$$

so that

$$\sin(x) - x\cos(x) > \frac{x}{2}[1-\cos(x)].$$

This refines the second item of Proposition 1.

4. Third Set of Trigonometric Inequalities

4.1. Statement

The statement below presents another new set of one-parameter trigonometric inequalities. The proof is based on the basic concave property (inequality) of the cosine and sine functions.

Proposition 4. *The integral method can be used to prove the classical inequalities below.*

1. *For any $\alpha \geq 0$ and $x \in (0, \pi/2)$, we have*

$$\int_0^1 u^\alpha \cos(ux)du > \frac{1}{\alpha+2}\left[\cos(x) + \frac{1}{\alpha+1}\right].$$

2. *For any $\alpha \geq 0$ and $x \in (0, \pi/2)$, we have*

$$\int_0^1 u^\alpha \sin(ux)du > \frac{1}{\alpha+2}\sin(x).$$

Proof of Proposition 4.

1. For any $u \in (0,1)$, $\alpha \geq 0$ and $x \in (0, \pi/2)$, $\cos(ux)$ is strictly concave with respect to u. The basic concave inequality gives

$$\cos(ux) = \cos[ux + (1-u)0] > u\cos(x) + (1-u)\cos(0) = u\cos(x) + (1-u).$$

We therefore have

$$\begin{aligned}
\int_0^1 u^\alpha \cos(ux)du > \int_0^1 u^\alpha \times [u\cos(x) + (1-u)]\, du &= \cos(x)\int_0^1 u^{\alpha+1}du + \int_0^1 u^\alpha du - \int_0^1 u^{\alpha+1}du \\
&= \cos(x) \times \frac{1}{\alpha+2} + \frac{1}{\alpha+1} - \frac{1}{\alpha+2} \\
&= \frac{1}{\alpha+2}\left[\cos(x) + \frac{1}{\alpha+1}\right].
\end{aligned}$$

2. For any $u \in (0,1)$, $\alpha \geq 0$ and $x \in (0, \pi/2)$, $\sin(ux)$ is strictly concave with respect to u. This implies that

$$\sin(ux) = \sin[ux + (1-u)0] > u\sin(x) + (1-u)\sin(0) = u\sin(x) + 0 = u\sin(x).$$

We therefore have

$$\int_0^1 u^\alpha \sin(ux)du > \int_0^1 u^\alpha \times u\sin(x)du = \sin(x)\int_0^1 u^{\alpha+1}du = \frac{1}{\alpha+2}\sin(x).$$

This concludes the proof of Proposition 4. □

The approach of mixing concave properties and the integral method seems to be unexploited in the literature of trigonometric inequalities. In this sense, this proposition fills a gap.

Table 4 presents some special cases of interest of this proposition, focusing on the first integer values of α.

α	$\int_0^1 u^\alpha \cos(ux)du$	$> \frac{1}{\alpha+2}\left[\cos(x)+\frac{1}{\alpha+1}\right]$
0	$\frac{\sin(x)}{x}$	$> \frac{1}{2}\left[\cos(x)+1\right]$
1	$\frac{x\sin(x)+\cos(x)-1}{x^2}$	$> \frac{1}{3}\left[\cos(x)+\frac{1}{2}\right]$
2	$\frac{(x^2-2)\sin(x)+2x\cos(x)}{x^3}$	$> \frac{1}{4}\left[\cos(x)+\frac{1}{3}\right]$
3	$\frac{x(x^2-6)\sin(x)+3(x^2-2)\cos(x)+6}{x^4}$	$> \frac{1}{5}\left[\cos(x)+\frac{1}{4}\right]$
4	$\frac{4x(x^2-6)\cos(x)+(x^4-12x^2+24)\sin(x)}{x^5}$	$> \frac{1}{6}\left[\cos(x)+\frac{1}{5}\right]$
α	$\int_0^1 u^\alpha \sin(ux)du$	$> \frac{1}{\alpha+2}\sin(x)$
0	$\frac{1-\cos(x)}{x}$	$> \frac{1}{2}\sin(x)$
1	$\frac{\sin(x)-x\cos(x)}{x^2}$	$> \frac{1}{3}\sin(x)$
2	$\frac{(2-x^2)\cos(x)+2x\sin(x)-2}{x^3}$	$> \frac{1}{4}\sin(x)$
3	$\frac{3(x^2-2)\sin(x)-x(x^2-6)\cos(x)}{x^4}$	$> \frac{1}{5}\sin(x)$
4	$\frac{4x(x^2-6)\sin(x)-(x^4-12x^2+24)\cos(x)+24}{x^5}$	$> \frac{1}{6}\sin(x)$

Table 4. Some special cases of interest of Proposition 4

To the best of our knowledge, the inequalities obtained for $\alpha = 1, 2, 3, 4$, as well as those associated with all values of $\alpha \geq 1$, are indeed new in the literature.

4.2. Focus on the cases $\alpha = 0$ and $\alpha = 1$

Applying the first item of Proposition 4 to $\alpha = 0$, for any $x \in (0, \pi/2)$, we get

$$\frac{\sin(x)}{x} > \frac{1}{2}\left[\cos(x)+1\right],$$

so that, using a classical trigonometric formula,

$$\frac{\sin(x)}{x} > \cos^2\left(\frac{x}{2}\right).$$

The second item gives

$$\frac{1-\cos(x)}{x} > \frac{1}{2}\sin(x),$$

so that, using a classical trigonometric formula,

$$\cos^2\left(\frac{x}{2}\right) > \frac{1}{4}x\sin(x).$$

Applying the first item of Proposition 4 to $\alpha = 1$, for any $x \in (0, \pi/2)$, we get

$$\frac{x\sin(x)+\cos(x)-1}{x^2} > \frac{1}{3}\left[\cos(x)+\frac{1}{2}\right].$$

The second item gives

$$\frac{\sin(x) - x\cos(x)}{x^2} > \frac{1}{3}\sin(x),$$

from which we derive

$$\sin(x) - x\cos(x) > \frac{1}{3}x^2\sin(x). \tag{3}$$

This inequality is sharper than the left-hand side inequality in Equation (2), and also implies the second item of Proposition 1.

5. Fourth Set of Trigonometric Inequalities

5.1. Statement

The statement below presents another new set of one-parameter trigonometric inequalities. The proof relies on the concave property of the cosine and sine functions, and the Jensen integral inequality, which is recalled in full generality in Appendix.

Proposition 5. *The integral method can be used to prove the trigonometric inequalities below.*

1. *For any $\alpha \geq 0$ and $x \in (0, \pi/2)$, we have*

$$\int_0^1 u^\alpha \cos(ux)du < \frac{1}{\alpha+1}\cos\left(\frac{\alpha+1}{\alpha+2}x\right).$$

2. *For any $\alpha \geq 0$ and $x \in (0, \pi/2)$, we have*

$$\int_0^1 u^\alpha \sin(ux)du < \frac{1}{\alpha+1}\sin\left(\frac{\alpha+1}{\alpha+2}x\right).$$

Proof of Proposition 5.

1. For any $u \in (0, 1)$, $\alpha \geq 0$ and $x \in (0, \pi/2)$, $\cos(ux)$ is strictly concave with respect to u. Furthermore, the function $p(u) = (\alpha+1)u^\alpha$ is a valid probability density function, i.e., it satisfies $p(u) \geq 0$ for any $u \in (0, 1)$ and $\int_0^1 p(u)du = 1$. This implies that

$$\int_0^1 u^\alpha \cos(ux)du = \frac{1}{\alpha+1}\int_0^1 p(u)\cos(ux)du < \frac{1}{\alpha+1}\cos\left(\int_0^1 p(u)uxdu\right)$$
$$= \frac{1}{\alpha+1}\cos\left((\alpha+1)x\int_0^1 u^{\alpha+1}du\right) = \frac{1}{\alpha+1}\cos\left(\frac{\alpha+1}{\alpha+2}x\right).$$

2. Similarly, but using the concavity of $\sin(ux)$ instead of that of $\cos(ux)$, we get

$$\int_0^1 u^\alpha \sin(ux)du = \frac{1}{\alpha+1}\int_0^1 p(u)\sin(ux)du < \frac{1}{\alpha+1}\sin\left(\int_0^1 p(u)uxdu\right)$$
$$= \frac{1}{\alpha+1}\sin\left((\alpha+1)x\int_0^1 u^{\alpha+1}du\right) = \frac{1}{\alpha+1}\sin\left(\frac{\alpha+1}{\alpha+2}x\right).$$

This ends the proof of Proposition 5. □

Table 5 presents some special cases of interest of this proposition, focusing on the first integer values of α.

α	$\int_0^1 u^\alpha \cos(ux)du$	$< \frac{1}{\alpha+1}\cos\left(\frac{\alpha+1}{\alpha+2}x\right)$
0	$\frac{\sin(x)}{x}$	$< \cos\left(\frac{1}{2}x\right)$
1	$\frac{x\sin(x)+\cos(x)-1}{x^2}$	$< \frac{1}{2}\cos\left(\frac{2}{3}x\right)$
2	$\frac{(x^2-2)\sin(x)+2x\cos(x)}{x^3}$	$< \frac{1}{3}\cos\left(\frac{3}{4}x\right)$
3	$\frac{x(x^2-6)\sin(x)+3(x^2-2)\cos(x)+6}{x^4}$	$< \frac{1}{4}\cos\left(\frac{4}{5}x\right)$
4	$\frac{4x(x^2-6)\cos(x)+(x^4-12x^2+24)\sin(x)}{x^5}$	$< \frac{1}{5}\cos\left(\frac{5}{6}x\right)$
α	$\int_0^1 u^\alpha \sin(ux)du$	$< \frac{1}{\alpha+1}\sin\left(\frac{\alpha+1}{\alpha+2}x\right)$
0	$\frac{1-\cos(x)}{x}$	$< \sin\left(\frac{1}{2}x\right)$
1	$\frac{\sin(x)-x\cos(x)}{x^2}$	$< \frac{1}{2}\sin\left(\frac{2}{3}x\right)$
2	$\frac{(2-x^2)\cos(x)+2x\sin(x)-2}{x^3}$	$< \frac{1}{3}\sin\left(\frac{3}{4}x\right)$
3	$\frac{3(x^2-2)\sin(x)-x(x^2-6)\cos(x)}{x^4}$	$< \frac{1}{4}\sin\left(\frac{4}{5}x\right)$
4	$\frac{4x(x^2-6)\sin(x)-(x^4-12x^2+24)\cos(x)+24}{x^5}$	$< \frac{1}{5}\sin\left(\frac{5}{6}x\right)$

Table 5. Some special cases of interest of Proposition 5

To the best of our knowledge, these inequalities are new in the literature, as well as those obtained for any $\alpha > 0$.

5.2. Focus on the cases $\alpha = 0$ and $\alpha = 1$

Applying the first item of Proposition 5 to $\alpha = 0$, for any $x \in (0, \pi/2)$, we get

$$\frac{\sin(x)}{x} < \cos\left(\frac{1}{2}x\right),$$

from which we derive

$$\sin(x) < x\cos\left(\frac{1}{2}x\right).$$

The second item gives

$$\frac{1-\cos(x)}{x} < \sin\left(\frac{1}{2}x\right).$$

Applying the first item of Proposition 5 to $\alpha = 1$, for any $x \in (0, \pi/2)$, we obtain

$$\frac{x\sin(x)+\cos(x)-1}{x^2} < \frac{1}{2}\cos\left(\frac{2}{3}x\right).$$

The second item gives

$$\frac{\sin(x)-x\cos(x)}{x^2} < \frac{1}{2}\sin\left(\frac{2}{3}x\right),$$

from which we derive

$$\sin(x) - x\cos(x) < \frac{1}{2}x^2 \sin\left(\frac{2}{3}x\right).$$

To the best of our knowledge, it is new in the literature. Combining this with Equation (3), for any $x \in (0, \pi/2)$, we obtain the following two-sided inequality with bounds of comparable nature:

$$\frac{1}{3}x^2 \sin(x) < \sin(x) - x\cos(x) < \frac{1}{2}x^2 \sin\left(\frac{2}{3}x\right). \tag{4}$$

The sharpness of the lower bound is illustrated in Figure 1. We see that the difference is very small for $x \in (0, 0.5)$.

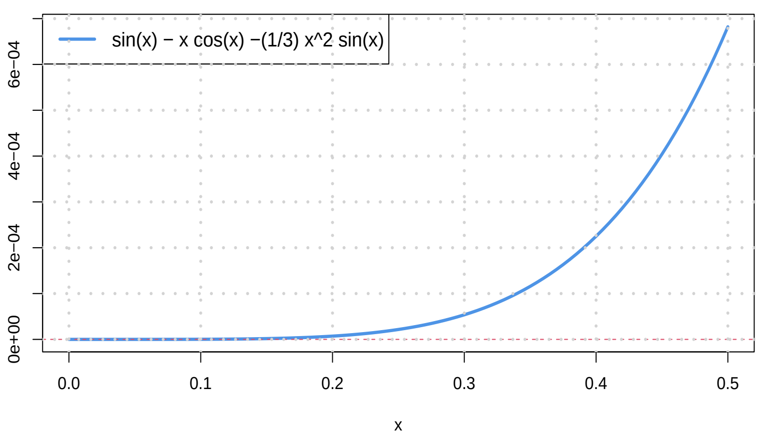

Fig. 1. Illustration of the sharpness of the lower bound in Equation (4) for $x \in (0, 0.5)$

The sharpness of the upper bound is illustrated in Figure 2. We also observe a difference which is very small for $x \in (0, 0.5)$.

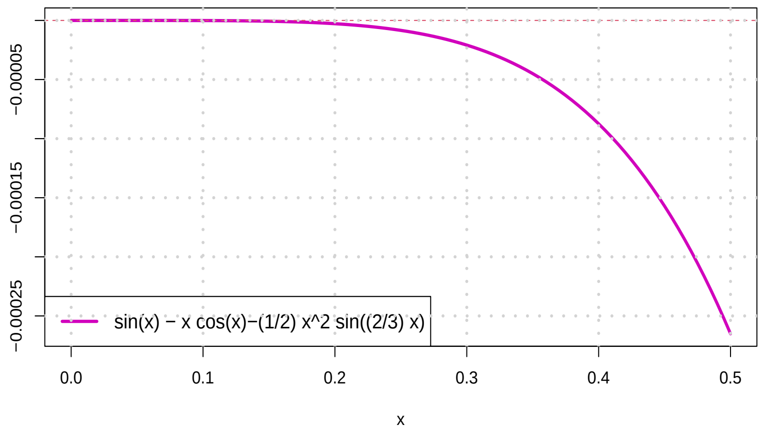

Fig. 2. Illustration of the sharpness of the upper bound in Equation (4) for $x \in (0, 0.5)$

These figures illustrate the interest of our findings.

6. Specific Set of Inequalities

The proposition below presents some specific trigonometric inequalities. The proof is based on double integral representations of the sine and cosine functions, and on Proposition 1. Some known results are recovered and new ones are presented.

Proposition 6. *The integral method can be used to prove the trigonometric inequalities below.*

1. *For any $x \in (0, \pi/2)$, we have*

$$\sin(x) > x\left(1 - \frac{1}{2}x\right).$$

2. *For any $x \in (0, \pi/2)$, we have*

$$\sin(x) > x\left(1 - \frac{1}{6}x^2\right).$$

3. *For any $x \in (0, \pi/2)$, we have*

$$\frac{\sin(x)}{x} < \frac{\cos(x) + 2}{3}.$$

4. *For any $x \in (0, \pi/2)$, we have*

$$\sin(x) < x\left(1 - \frac{1}{3\pi}x^2\right).$$

5. *For any $x \in (0, \pi/2)$, we have*

$$\cos(x) > 1 - \frac{1}{2}x^2.$$

6. *For any $x \in (0, \pi/2)$, we have*

$$\cos(x) < 1 - \frac{1}{2}x^2 + \frac{1}{3\pi}x^3.$$

7. *For any $x \in (0, \pi/2)$, we have*

$$\sin(x) > \frac{1}{5}\left[x\cos(x) - 4x + \frac{1}{3}x^3\right].$$

Proof of Proposition 6.

1. It follows from two successive integral developments that

$$\sin(x) = x\int_0^1 \cos(tx)dt = x\int_0^1\int_0^1 [1 - tx\sin(tux)]dudt.$$

For any $t, u \in (0, 1)$, $\alpha \geq 0$ and $x \in (0, \pi/2)$, we have $\sin(tux) < 1$. This and another basic integral calculus yield

$$\sin(x) > x\int_0^1\int_0^1 (1 - tx \times 1)dudt = x\int_0^1\int_0^1 (1 - tx)dudt = x\left(1 - \frac{1}{2}x\right).$$

2. We use again

$$\sin(x) = x\int_0^1\int_0^1 [1 - tx\sin(tux)]dudt.$$

For any $t, u \in (0, 1)$, $\alpha \geq 0$ and $x \in (0, \pi/2)$, the first item of Proposition 1 ensures that $\sin(tux) < tux$. This and a simple integral calculus give

$$\sin(x) > x\int_0^1\int_0^1 [1 - tx \times (tux)]dudt = x\int_0^1\int_0^1 (1 - t^2x^2u)dudt = x\left(1 - \frac{1}{6}x^2\right).$$

3. We consider again

$$\sin(x) = x\int_0^1\int_0^1 [1 - tx\sin(tux)]dudt.$$

For any $t, u \in (0, 1)$, $\alpha \geq 0$ and $x \in (0, \pi/2)$, using the second item of Proposition 1, we have $\sin(tux) > tux\cos(tux)$. This and a basic integral calculus yield

$$\sin(x) < x\int_0^1\int_0^1 [1 - tx \times tux\cos(tux)]dudt = x\int_0^1\int_0^1 [1 - t^2x^2u\cos(tux)]dudt = x\left[-2\frac{\sin(x)}{x} + \cos(x) + 2\right],$$

so that

$$\frac{\sin(x)}{x} < -2\frac{\sin(x)}{x} + \cos(x) + 2$$

and

$$\frac{\sin(x)}{x} < \frac{\cos(x) + 2}{3}.$$

4. We use again

$$\sin(x) = x\int_0^1\int_0^1 [1 - tx\sin(tux)]dudt.$$

For any $t, u \in (0, 1)$, $\alpha \geq 0$ and $x \in (0, \pi/2)$, the third item of Proposition 1 ensures that $\sin(tux) > (2/\pi)tux$. This and a simple integral calculus yield

$$\sin(x) < x\int_0^1\int_0^1 \left[1 - tx \times \frac{2}{\pi}(tux)\right] dudt = x\int_0^1\int_0^1 \left(1 - \frac{2}{\pi}t^2x^2u\right) dudt = x\left(1 - \frac{1}{3\pi}x^2\right).$$

5. A double integral development yields

$$\cos(x) = \int_0^1 [1 - x\sin(tx)]dt = \int_0^1 \left[1 - x(tx)\int_0^1 \cos(tux)du\right] dt = \int_0^1 \int_0^1 \left[1 - tx^2\cos(tux)\right] dudt.$$

For any $t, u \in (0,1)$, $\alpha \geq 0$ and $x \in (0, \pi/2)$, we have $\cos(tux) < 1$. This and a basic integral calculus give

$$\cos(x) > \int_0^1 \int_0^1 \left(1 - x^2 t\right) dudt = 1 - \frac{1}{2}x^2.$$

6. We use again

$$\cos(x) = \int_0^1 \int_0^1 \left[1 - tx^2\cos(tux)\right] dudt.$$

For any $t, u \in (0,1)$, $\alpha \geq 0$ and $x \in (0, \pi/2)$, the fourth item of Proposition 1 ensures that $\cos(tux) > 1 - (2/\pi)tux$. This and a basic integral calculus give

$$\cos(x) < \int_0^1 \int_0^1 \left[1 - x^2 t\left(1 - \frac{2}{\pi}tux\right)\right] dudt = 1 - \frac{1}{2}x^2 + \frac{1}{3\pi}x^3.$$

7. We use again

$$\sin(x) = x\int_0^1 \int_0^1 [1 - tx\sin(tux)]dudt.$$

For any $t, u \in (0,1)$, $\alpha \geq 0$ and $x \in (0, \pi/2)$, based on the third item of this present proposition, we have $\sin(tux) < tux[\cos(tux) + 2]/3$. This and an integral development give

$$\begin{aligned}
\sin(x) &> x\int_0^1 \int_0^1 (1 - t^2x^2u\left[\frac{\cos(tux) + 2}{3}\right])dudt \\
&= x\left\{-\frac{1}{9x}[x^3 - 12x + 6\sin(x) - 3x\cos(x)]\right\} \\
&= -\frac{1}{9}[x^3 - 12x + 6\sin(x) - 3x\cos(x)],
\end{aligned}$$

so that

$$\frac{5}{3}\sin(x) > -\frac{1}{9}[x^3 - 12x - 3x\cos(x)]$$

and

$$\sin(x) > \frac{1}{5}\left[x\cos(x) - 4x + \frac{1}{3}x^3\right].$$

This concludes the proof of Proposition 6. □

These items can also be seen as refinements of the main inequalities in Proposition 1.

The fourth item corresponds to the Cusa-Huygens inequality. The fifth and sixth items yield the following two-sided inequality:

$$1 - \frac{1}{2}x^2 < \cos(x) < 1 - \frac{1}{2}x^2 + \frac{1}{3\pi}x^3, \tag{5}$$

which is new to our knowledge. The sharpness of the lower bound is illustrated in Figure 3. The difference is very small for $x \in (0, 0.5)$.

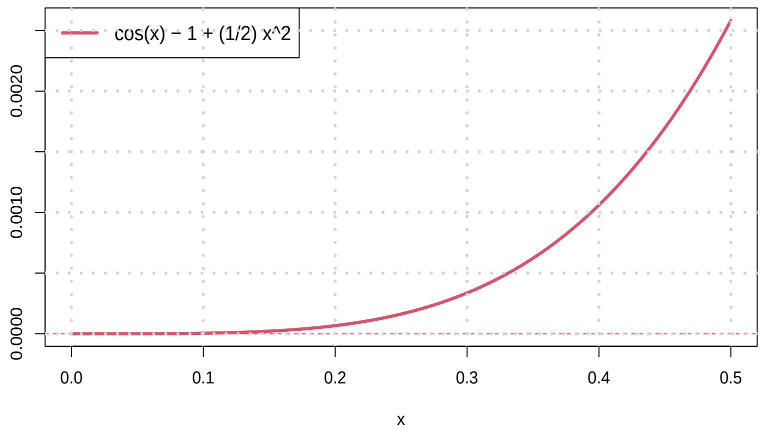

Fig. 3. Illustration of the sharpness of the lower bound in Equation (5) for $x \in (0, 0.5)$

The sharpness of the upper bound is illustrated in Figure 4. We also observe a difference which is very small for $x \in (0, 0.5)$.

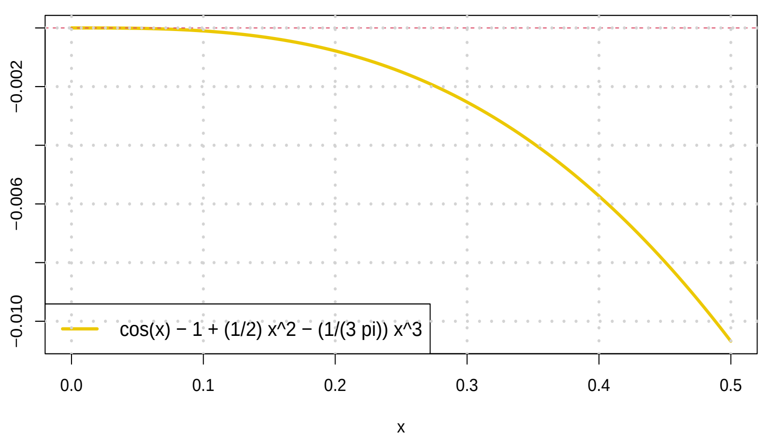

Fig. 4. Illustration of the sharpness of the upper bound in Equation (5) for $x \in (0, 0.5)$

These figures illustrate the interest of our findings.

7. Conclusion

In this article, we have examined the use of integral methods to prove trigonometric inequalities. While such methods are often overlooked, we have shown that they are both effective and versatile. Using basic tools such as primitive integration, the Chebyshev integral inequality and the Jensen integral inequality, we have recovered classical results and derived new ones. All results are proved in detail. The methods are accessible and suitable for use in teaching, thus contributing to mathematical education. They also provide new directions for research. Future work may involve extending these techniques to other classes of functions, such as hyperbolic or inverse trigonometric functions. Another possible perspective is to use them to deepen the notion of trigonometrically convex function, as developed in [7]. One could also investigate inequalities involving integrals with parameters, or apply these ideas to numerical analysis, such as in the context of [16], special functions and mathematical physics. These deserve further exploration in the context of inequalities.

Appendix

The general statements of the Chebyshev integral inequality and the Jensen integral inequality are recalled below.

Chebyshev integral inequality

Let $a, b \in \mathbb{R}$ with $a < b$ and let $f, g : [a, b] \to [0, +\infty)$ be two integrable monotonic functions. If f and g are of opposite monotonicity, then the Chebyshev integral inequality states that

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \leq \left[\frac{1}{b-a}\int_a^b f(x)dx\right]\left[\frac{1}{b-a}\int_a^b g(x)dx\right].$$

If f and g are both increasing or both decreasing, then this inequality is reversed

Jensen integral inequality

Let I be an interval with $I \subseteq \mathbb{R}$, $\phi : I \to \mathbb{R}$ be a concave function, let $a, b \in \mathbb{R}$ with $a < b$ and let $f : [a, b] \to I$ be an integrable function. Let $p : [a, b] \to [0, +\infty)$ be a probability density function, i.e., such that $\int_a^b p(x)dx = 1$. Then the Jensen integral inequality states that

$$\int_a^b \phi(f(x))p(x)dx \leq \phi\left(\int_a^b f(x)p(x)dx\right)$$

If f is convex instead of concave, then this inequality is reversed

Equality holds if and only if f is constant almost everywhere or ϕ is affine on the convex hull of the range of f.

Declarations

Acknowledgements: The author would like to express his sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions

Author's Contribution: The author, C.C., contributed to this manuscript fully in theoretic and structural points.

Conflict of Interest Disclosure: The author declares no conflict of interest.

Copyright Statement: Author owns the copyright of their work published in the journal and their work is published under the CC BY-NC 4.0 license.

Supporting/Supporting Organizations: This research received no external funding.

Ethical Approval and Participant Consent: This article does not contain any studies with human or animal subjects. It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of Data and Materials: Data sharing not applicable.

Use of AI tools: The author declares that they have not used Artificial Intelligence (AI) tools in the creation of this article.

ORCID

Christophe Chesneau https://orcid.org/0000-0002-1522-9292

References

[1] D.S. Mitrinović, *Analytic Inequalities*, Springer-Verlag, Berlin, 1970. [Web]

[2] Y.J. Bagul, B. Banjac, C. Chesneau, M. Kostić and B. Malešević, *New refinements of Cusa-Huygens inequality,* Results Math., **76** (2021), 107. [CrossRef] [Scopus] [Web of Science]

[3] Y.J. Bagul, C. Chesneau and M. Kostić, *On the Cusa-Huygens inequality,* Rev. Real Acad. Cienc. Exactas Fis. Nat. - A: Mat., **115** (2021), 29. [CrossRef] [Scopus] [Web of Science]

[4] G. Bercu, *Fourier series method related to Wilker-Cusa-Huygens inequalities,* Math. Inequal. Appl., **22**(4) (2019), 1091–1098. [CrossRef] [Scopus] [Web of Science]

[5] B. Chaouchi, V.E. Fedorov and M. Kostić, *Monotonicity of certain classes of functions related with Cusa-Huygens inequality,* Chelyab. Fiz. Mat. Zh., **6**(3) (2021), 331–337. [CrossRef] [Scopus]

[6] C.P. Chen and W.S. Cheung, *Sharp Cusa and Becker-Stark inequalities*, J. Inequal. Appl., **2011**, 136, (2011). [CrossRef] [Scopus] [Web of Science]

[7] H. Kadakal, *Hermite-Hadamard type inequalities for trigonometrically convex functions*, Sci. Stud. Res., Ser. Math. Inform., **28**(2) (2018), 19-28. [Web]

[8] C. Huygens, *Oeuvres Completes*, Société Hollandaise des Sciences, Haga, 1888-1940.

[9] B. Malešević, M. Nenezić, L. Zhu, B. Banjac and M. Petrović, *Some new estimates of precision of Cusa-Huygens and Huygens approximations,* Appl. Anal. Discrete Math., **15**(1) (2021), 243–259. [CrossRef] [Scopus] [Web of Science]

[10] B. Malešević and M. Makragić, *A method for proving some inequalities on mixed trigonometric polynomial functions*, J. Math. Inequal., **10**(3) (2016), 849–876. [CrossRef] [Scopus] [Web of Science]

[11] C. Mortici, *The natural approach of Wilker-Cusa-Huygens Inequalities*, Math. Inequal. Appl., **14**(3) (2011), 535–541. [CrossRef] [Scopus] [Web of Science]

[12] J. Sándor, *Sharp Cusa-Huygens and related inequalities*, Notes Number Theory Discrete Math., **19**(1) (2013), 50–54. [Web] [Web of Science]

[13] J. Sándor and R. Oláh-Gal, *On Cusa-Huygens type trigonometric and hyperbolic inequalities*, Acta. Univ. Sapientiae Math., **4**(2) (2012), 145–153. [Web]

[14] Z.H. Yang, *Refinements of a two-sided inequality for trigonometric functions,* J. Math. Inequal., **7**(4) (2013), 601–615. [CrossRef] [Scopus] [Web of Science]

[15] L. Zhu, *New Cusa-Huygens type inequalities,* AIMS Math., **5**(5) (2020), 5320-5331. [CrossRef] [Scopus] [Web of Science]

[16] L.G. Molinari, *A note on trigonometric approximations of Bessel functions of the first kind, and trigonometric power sums*, Fundam. J. Math. Appl. **5**(4) (2022), 266-272. [CrossRef]

All open access articles published are distributed under the terms of the CC BY-NC 4.0 license (Creative Commons Attribution-Non-Commercial 4.0 International Public License as currently displayed at http://creativecommons.org/licenses/by-nc/4.0/legalcode) which permits unrestricted use, distribution, and reproduction in any medium, for non-commercial purposes, provided the original work is properly cited.

How to cite this article: C. Chesneau, *A collection of trigonometric inequalities using integral methods*, Adv. Anal. Appl. Math., **2**(1) (2025), 1-20. DOI 10.62298/advmath.22