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Research Paper

The Szeged Index & Energy of Commuting Graph of Certain Finite Non-commutative Groups

Subarsha Banerjee[1,*] and Ankita Agarwal[2,†]

[1]Department of Mathematics, JIS University, 81 Nilgunj Road, Agarpara, West Bengal, India
[2]Department of Computer Science & Engineering, Nilgunj Road, Agarpara, West Bengal, India

*To whom correspondence should be addressed: subarsha.banerjee@jisuniversity.ac.in
† 12ankitaagarwal330@gmail.com

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Abstract

Let G be a finite group having center $Z(G)$. The commuting graph of G denoted by $\mathcal{C}(G)$ has vertex set as $G \setminus Z(G)$, and two vertices x and y are adjacent in $\mathcal{C}(G)$ if x and y commute with each other. The commuting graph of a finite group is a powerful tool in group theory to understand the internal structure of the group. Through its graph-theoretic properties, the commuting graph helps in classifying groups and understanding how the group works internally. It serves as a visual and computational tool to complement other algebraic methods in group theory. The Szeged index is a topological index used in the study of molecular structures, particularly in chemistry and chemical graph theory. It is a numerical value that characterizes the connectivity of a molecular graph. In this paper, we have determined the Szeged index of the commuting graph of various finite non-commutative groups, namely the dihedral group D_n, and the dicyclic group Dic_n. Moreover, we determine the energy of $\mathcal{C}(\mathrm{D}_n)$ and $\mathcal{C}(\mathrm{Dic}_n)$. A graph is said to be hyperenergetic if the energy of G is greater than the complete graph. In this paper, we prove that the graphs $\mathcal{C}(\mathrm{D}_n)$ and $\mathcal{C}(\mathrm{Dic}_n)$ are non-hyperenergetic graphs.

Key Words: Commuting Graph, Non-Commutative Group, Szeged Index, Energy, Adjacency Matrix, Eigenvalues.
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1. Introduction

In theoretical chemistry, a topological index, also known as the connectivity index is used to study the molecular structure of a chemical compound. There are various types of topological indices which are categorized based on their degree, distance, and various other graph invariants. The topological index of a graph that is the oldest and thoroughly examined is the Wiener index which is based on the distance between the vertices of a given graph, see [1, 2] for some references. Two well-known and well-studied degree-based topological indices are the 1st Zagreb index [3] and the 2nd Zagreb index [4]. The Szeged index of a graph has been introduced much later and it has been studied in great detail nowadays. The Szeged index has a close association with the Wiener index. Introduced by Ivan Gutman, the Szeged index generalizes the concept of the Wiener index. A few basic mathematical properties of the Szeged index have been derived in [5, 6], and its certain chemical applications have been studied in [7, 8].

We shall assume $\mathcal{G}$ to be a *simple* and a *connected* graph. Let $V(\mathcal{G})$ and $E(\mathcal{G})$ denote the vertex set and the edge set of $\mathcal{G}$ respectively. Let $\mid \mathcal{G} \mid$ denote the *order* of $\mathcal{G}$. Two *adjacent* vertices a and b are denoted by $a \sim b$. The *join* of two graphs $\mathcal{G}_1 = (V_1, E_1)$ and $\mathcal{G}_2 = (V_2, E_2)$ is denoted by $\mathcal{G}_1 \vee \mathcal{G}_2$, and the *union* of two graphs is denoted by $\mathcal{G}_1 \cup \mathcal{G}_2$. We follow [9] for basic definitions in graph theory. The distance between two vertices $a, b \in \mathcal{G}$, denoted by $d(a, b)$, is defined to be the length of the shortest path from a to b. The *Wiener index* [10] of a connected graph $\mathcal{G}$ is defined as follows:

$$W(\mathcal{G}) = \frac{1}{2} \sum_{a,b \in V(\mathcal{G})} d(a, b). \tag{1}$$

Suppose $e \in E(\mathcal{G})$ is an edge between the vertices a and b of $\mathcal{G}$. We first state the following definitions: We define

$$\begin{aligned} n_1(ab|\mathcal{G}) &= |\{x \in \mathcal{G} : d(x, a) < d(x, b)\}| \\ n_2(ab|\mathcal{G}) &= |\{x \in \mathcal{G} : d(x, b) < d(x, a)\}|. \end{aligned} \tag{2}$$

The two quantities described in (2) were mentioned for the first time in [11]. For a long time, it was known that the formula

$$W(\mathcal{G}) = \sum_{e(=ab) \in E(\mathcal{G})} n_1(ab|\mathcal{G}) n_2(ab|\mathcal{G}) \tag{3}$$

holds for molecular graphs of alkanes. In [12], it was proved that (3) holds for all trees. Furthermore, in [5] it was shown that (3) does not hold in general (in particular for graphs containing cycles), and only holds for graphs whose each block is a complete graph. Although the attempts to change the right-hand side of (3) to make it applicable to all connected graphs have been successfully made in [13, 14], the resulting expressions were very confusing. In [6], it was suggested that the complications arising with the generalization of (3) to all connected graphs containing cycles could be overcome by using the right–hand side of (3) as the definition of a new graph invariant. Consequently, the formula was extended to all graphs and it came to be known as the Szeged index of a graph. The *Szeged index* of a connected graph $\mathcal{G}$ is defined as follows:

$$Sz(\mathcal{G}) = \sum_{e(=ab) \in E(\mathcal{G})} n_1(ab|\mathcal{G}) n_2(ab|\mathcal{G}). \tag{4}$$

The Szeged index has been considered from multiple viewpoints; see, for example, [15, 16] and the references therein for some literature on the same. The Szeged index of the Cartesian product of graphs([10]), join and composition of graphs([17]), and bridge graphs([18]) have been determined by various authors in the recent past. The Szeged index of generalized join of graphs has been studied in [19]. Recently, the Szeged index of unicyclic graphs was studied in [20].

Given a finite group G, the *commuting graph* of G, $\mathcal{C}(G)$, has vertex set as $V = G \setminus Z(G)$ and any two vertices $x, y \in V$ are adjacent only if $xy = yx$. The study of the *commuting graph* of a finite group provides valuable insights into the structure and properties of the group. By encoding the commutative relationships between its elements, the commuting graph reveals hidden interactions that might not be immediately obvious through other methods of group analysis. There are several key motivations for studying the commuting graph, ranging from understanding the group structure to gaining insights into specific classes of groups. The commuting graph reveals the *centralizer structure* of the group. The centralizer of an element $g \in G$, denoted $C_G(g)$, is the set of all elements in G that commute with g. The commuting graph helps us visualize how elements interact by commuting with each other, and the induced subgraph corresponding to a centralizer will be a *complete subgraph*. By studying these induced subgraphs, we can gain deeper insights into the group's internal structure, such as which elements share the same centralizer and how centralizers relate to other subgroups. The *connectivity* of the commuting graph provides important information about the internal structure of the group. For example, if the graph is highly connected, it might indicate that many elements in the group share centralizing properties. If the graph is disconnected, it suggests that the group has a more complex structure with isolated parts (such as distinct conjugacy classes or normal subgroups). The *diameter* of the graph (the longest shortest path between any two vertices) can also offer insights into the group's overall complexity and how the elements are related in terms of commutation. The center of the group forms a *clique* in the commuting graph. If the group has a large center or centralizers, these cliques can be substantial, revealing the abelian or near-abelian portions of the group. Identifying these cliques helps in understanding the balance between abelian and non-abelian parts of the group. The *chromatic number* of the commuting graph (the minimum number of colors needed to color the graph so that no two adjacent vertices share the same color) can provide insight into the structure of conjugacy classes or the dynamics of group actions. This can be particularly useful in understanding how the group can be decomposed or classified based on the interactions of its elements.

The *non-commuting graph* of G, $\mathcal{NC}(G)$, has vertex set as $V = G \setminus Z(G)$ and any two vertices $x, y \in V$ are adjacent only if $xy \neq yx$. Here, $Z(G)$ denotes the *center* of the group G, i.e.

$$Z(G) = \{x \in G : xg = gx \text{ forall } g \in G\}.$$

Chelvam *et al.* in [21] considered the commuting graph of the dihedral group D_n, and studied its various properties. In [22], the authors studied the distant properties as well as detour distant properties of $\mathcal{C}(\mathrm{D}_n)$. The metric dimension and resolving polynomial of $\mathcal{C}(\mathrm{D}_n)$ have also been studied by them. The commuting graph of D_n was also studied in [23]. The spectral properties of commuting graphs of various finite groups have been studied in [24, 25]. The metric dimension of the commuting graph of generalized dihedral groups has been studied in [26]. The metric dimension and the resolving polynomial of the non-commuting graph of D_n was studied in [27].

It has been an active research topic over the past few years to study the topological properties of commuting and non-commuting graphs of various finite groups [28, 29]. The Szeged index of commuting graph has not been determined yet, which puts a gap in the literature. This motivates us in this paper to investigate the Szeged index of the commuting graph of the dihedral group D_n, and the dicyclic group Dic_n.

The *energy* $\mathcal{E}(\mathcal{G})$ of a graph $\mathcal{G}$ is defined to be the sum of absolute values of all the eigenvalues of the adjacency matrix of $\mathcal{G}$. Since the eigenvalues of the adjacency matrix of the complete graph, K_n are $n-1$ having multiplicity 1, and -1 having multiplicity $n-1$, the energy of the complete graph is $2(n-1)$. We say a graph G to be *hyperenergetic* if $\mathcal{E}(\mathcal{G}) > \mathcal{E}(K_n) = 2(n-1)$, and non-hyperenergetic if $\mathcal{E}(\mathcal{G}) < \mathcal{E}(K_n) = 2(n-1)$. In [30], it was shown that the line graph of K_n is hyperenergetic for $n \geq 5$. Nikiforv obtained a significant result regarding hyperenergetic graphs in [31]. The author had shown that for almost all graphs $\mathcal{E}(\mathcal{G}) = \left(\frac{4}{3\pi} + o(1)\right)n^{\frac{3}{2}}$, which implies that almost all graphs are hyperenergetic. Consequently, the problem of finding non-hyperenergetic graphs is quite significant, see [32]. In this paper, we prove that the graphs $\mathcal{C}(\mathrm{D}_n)$ and $\mathcal{C}(\mathrm{Dic}_n)$ are not hyperenergetic for any n.

The paper has been arranged as follows: In (2), we compute the Szeged index of $\mathcal{C}(\mathrm{D}_n)$ and $\mathcal{C}(\mathrm{Dic}_n)$. In (3), we prove that $\mathcal{C}(\mathrm{D}_n)$ and $\mathcal{C}(\mathrm{Dic}_n)$ are non-hyperenergetic for all n.

2. Szeged Index of Commuting Graph of Non-commutative Groups

In this section, we shall compute the Szeged index of commuting graph of non-commutative groups like the dihedral group D_n and the dicyclic group Dic_n. We first determine the Szeged index of commuting graph of the dihedral group D_n.

2.1. Szeged Index of Commuting Graph of Dihedral group D_n

The dihedral group D_n of order $2n$ has the following representation:

$$\mathrm{D}_n = \langle r, s : r^n = s^2 = 1, rs = sr^{-1} \rangle.$$

Moreover, the center of D_n is given as follows:

$$Z(\mathrm{D}_n) = \begin{cases} \{1, r^{\frac{n}{2}}\} & \text{if } n \text{ is even,} \\ \{1\} & \text{if } n \text{ is odd.} \end{cases}$$

We now provide the two main results of this section, viz. (1), (2).

Theorem 1. *If n is odd, then the Szeged index of $\mathcal{C}(\mathrm{D}_n)$ is given as follows:*

$$Sz(\mathcal{C}(\mathrm{D}_n)) = \frac{n(7n-5)}{2}.$$

Proof. Since n is odd, so using [22, Proposition 2.2], we have

$$\mathcal{C}(\mathrm{D}_n) = K_1 \vee (K_{n-1} \cup \overline{K}_n).$$

Now, if e is an edge of $\mathcal{C}(\mathrm{D}_n)$, then we have the following possibilities:
Case 1: $e = ab$ where $a \in K_1$ and $b \in K_{n-1} \cup \overline{K}_n$ or vice versa.

Case 2: $e = ab$ where $a, b \in K_{n-1}$.

We shall now consider the two cases given above one by one in what follows.
Case 1: Let $e = ab$ where $a \in K_1$ and $b \in K_{n-1} \cup \overline{K}_n$.
Since $b \in K_{n-1} \cup \overline{K}_n$, so we again have the following two cases:
Subcase 1: $b \in K_{n-1}$. Let v be a vertex of $\mathcal{C}(\mathrm{D}_n)$ such that $v \notin \{a, b\}$. Now, if $v \in K_{n-1}$, then $d(v, a) = d(v, b) = 1$. If $v \in \overline{K}_n$, then $d(v, a) = 1$, and $d(v, b) = 2$. Since, $n_1(ab|\mathcal{C}(\mathrm{D}_n)) = n+1, n_2(ab|\mathcal{C}(\mathrm{D}_n)) = 1$, so $n(ab|\mathcal{C}(\mathrm{D}_n)) = n+1$. Since there exist $n-1$ edges of the form ab where $a \in K_1$ and $b \in K_{n-1}$, so

$$\sum_{\substack{e(=ab)\in\mathcal{C}(\mathrm{D}_n)\\ a\in K_1 \text{ and } b\in K_{n-1}}} n(ab|\mathcal{C}(\mathrm{D}_n)) = n^2 - 1. \tag{5}$$

Subcase 2: $b \in \overline{K}_n$. Let v be a vertex of $\mathcal{C}(\mathrm{D}_n)$ such that $v \notin \{a, b\}$. Now, if $v \in K_{n-1}$, then $d(v, a) = 1$, and $d(v, b) = 2$. If $v \in \overline{K}_n$, then $d(v, a) = 1$, and $d(v, b) = 2$. Since $\overline{K}_{n-1}$ has $n-1$ vertices, and $\overline{K}_n$ has $n-1$ vertices other than b, so $n_1(ab|\mathcal{C}(\mathrm{D}_n)) = 2n-1, n_2(ab|\mathcal{C}(\mathrm{D}_n)) = 1$. Consequently, $n(ab|\mathcal{C}(\mathrm{D}_n)) = n_1(ab|\mathcal{C}(\mathrm{D}_n))n_2(ab|\mathcal{C}(\mathrm{D}_n)) = 2n-1$. Since there exist n edges of the form ab where $a \in K_1$ and $b \in \overline{K}_n$, so

$$\sum_{e(=ab)\in\mathcal{C}(\mathrm{D}_n),\quad a\in K_1 \text{ and } b\in \overline{K}_n} n(ab|\mathcal{C}(\mathrm{D}_n)) = 2n^2 - n. \tag{6}$$

Case 2: Let $e = ab$ where $a, b \in K_{n-1}$.

Let v be a vertex of $\mathcal{C}(\mathrm{D}_n)$ such that $v \notin \{a, b\}$. Now, if $v \in K_{n-1}$, then $d(v, a) = d(v, b) = 1$. Also, if $v \in \overline{K}_n$, then $d(v, a) = d(v, b) = 2$. Moreover, if $v \in K_1$, then $d(v, a) = d(v, b) = 1$, so $n(ab|\mathcal{C}(\mathrm{D}_n)) = n_1(ab|\mathcal{C}(\mathrm{D}_n))n_2(ab|\mathcal{C}(\mathrm{D}_n)) = 1$. Since there exist $\binom{n-1}{2} = \frac{(n-1)(n-2)}{2}$ edges of the form ab where $a, b \in K_{n-1}$, so

$$\sum_{e(=ab)\in\mathcal{C}(\mathrm{D}_n),\quad a,b\in K_{n-1}} n(ab|\mathcal{C}(\mathrm{D}_n)) = \frac{(n-1)(n-2)}{2}. \tag{7}$$

Using (5), (6) and (7), we obtain

$$\begin{aligned} Sz(\mathcal{C}(\mathrm{D}_n)) &= \sum_{e(=ab)\in\mathcal{C}(\mathrm{D}_n)} n(ab|\mathcal{C}(\mathrm{D}_n)) \\ &= (n^2-1) + (2n^2-n) + \left(\frac{(n-1)(n-2)}{2}\right) \end{aligned}$$

$$= \frac{7n^2 - 5n}{2}.$$

□

Theorem 2. *If n is even, then the Szeged index of $\mathcal{C}(\mathrm{D}_n)$ is given as follows:*

$$Sz(\mathcal{C}(\mathrm{D}_n)) = \frac{13n^2}{2} - 10n.$$

Proof. Since n is even, using [22, Proposition 2.2], we have

$$\mathcal{C}(\mathrm{D}_n) = K_2 \vee (K_{n-2} \cup \frac{n}{2}K_2).$$

If e is an edge of $\mathcal{C}(\mathrm{D}_n)$, then we have the following cases:
Case 1: $e = ab$ where $a \in K_2$ and $b \in K_{n-2} \cup \frac{n}{2}K_2$ or vice versa,

Case 2: $e = ab$ where $a, b \in K_{n-2}$,

Case 3: $e = ab$ where $a, b \in (n/2)K_2$.,

Case 4: $e = ab$ where $a, b \in K_2$.

We shall now consider the four cases in the order listed above in what follows.
Case 1: Let $e = ab$ where $a \in K_2$ and $b \in K_{n-2} \cup \frac{n}{2}K_2$. Since $b \in K_{n-2} \cup \frac{n}{2}K_2$, so we again have the following two sub-cases:

Subcase 1: $b \in K_{n-2}$. Let v be a vertex of $\mathcal{C}(\mathrm{D}_n)$ such that $v \notin \{a, b\}$.
Since v is a vertex of $\mathcal{C}(\mathrm{D}_n)$, the following possibilities may arise, either $v \in K_2$, or $v \in K_{n-2}$, or $v \in \frac{n}{2}K_2$.
We shall now list down the distance of the vertex v from a and b in a tabular form in (1).

$v \in K_2$	$v \in K_{n-2}$	$v \in \frac{n}{2}K_2$
$d(v, a) = 1$	$d(v, a) = 1$	$d(v, a) = 1$
$d(v, b) = 1$	$d(v, b) = 1$	$d(v, b) = 2$

Table 1. Possible Distances of v from a and b

Using (1), we observe that $d(v, a) < d(v, b)$ only if $v \in \frac{n}{2}K_2$. So, $n_1(ab|\mathcal{C}(\mathrm{D}_n)) = 1$ and $n_2(ab|\mathcal{C}(\mathrm{D}_n)) = n + 1$. Consequently, $n(ab|\mathcal{C}(\mathrm{D}_n)) = n + 1$. We note that there exist $2(n - 2)$ edges of the form ab where $a \in K_2$ and $b \in K_{n-2}$. So,

$$\sum_{\substack{e(=ab) \in \mathcal{C}(\mathrm{D}_n) \\ a \in K_2 \text{ and } b \in K_{n-2}}} n(ab|\mathcal{C}(\mathrm{D}_n)) = 2(n - 2)(n + 1). \tag{8}$$

Subcase 2: $b \in (n/2)K_2$. Then b is a vertex of K_2 for some n.
Let v be a vertex of $\mathcal{C}(\mathrm{D}_n)$ such that $v \notin \{a, b\}$.
Similar to Subcase 1, the following possibilities may arise, either $v \in K_2$, or $v \in K_{n-2}$, or $v \in \frac{n}{2}K_2$,

$v \in K_2$	$v \in K_{n-2}$	$v \in \frac{n}{2}K_2$
$d(v,a) = 1$	$d(v,a) = 1$	$d(v,a) = 1$
$d(v,b) = 1$	$d(v,b) = 2$	$d(v,b) = 1$ or 2

Table 2. Possible Distances of v from a and b

We shall now list down the distance of the vertex v from the vertices a and b in a tabular form in the following table((2)):

We note that if $v \in (n/2)K_2$, then $d(v,b) = 1$ or $d(v,b) = 2$. We shall explain the above fact in detail in what follows.

If $v \in (n/2)K_2$, then either b and v are in the same component of $(n/2)K_2$ or they lie in different components of $(n/2)K_2$. If b and v lie in the same component of $(n/2)K_2$, then we must have $d(v,b) = 1$.

Now, if b and v lie in different components of $(n/2)K_2$, then we must have $d(v,b) = 2$. We observe that there are $(\frac{n}{2} - 1)$ such components of $(n/2)K_2$ where v may lie for which $d(v,b) = 2$. Hence we get $2(\frac{n}{2} - 1)$ such vertices in $(n/2)K_2$ for which $d(v,a) < d(v,b)$. Moreover, we also have $n-2$ vertices of K_{n-2} for which $d(v,a) < d(v,b)$. So, $n_1(ab|\mathcal{C}(\mathrm{D}_n)) = 2n - 3$, and $n_2(ab|\mathcal{C}(\mathrm{D}_n)) = 1$, This gives, $n(ab|\mathcal{C}(\mathrm{D}_n)) = 2n - 3$. Since there are $2n$ edges of the form ab where $a \in K_2$ and $b \in \frac{n}{2}K_2$, we get

$$\sum_{e(=ab)\in\mathcal{C}(\mathrm{D}_n),\quad a\in K_2 \text{ and } b\in\frac{n}{2}K_2} n(ab|\mathcal{C}(\mathrm{D}_n)) = 2(2n^2 - 3n). \tag{9}$$

Case 2: Let $e = ab$ where $a, b \in K_{n-2}$.

Let v be a vertex of $\mathcal{C}(\mathrm{D}_n)$ such that $v \notin \{a, b\}$. Now, if $v \in K_{n-2}$, then $d(v,a) = d(v,b) = 1$. Also, if $v \in K_2$, then $d(v,a) = d(v,b) = 1$. Moreover, if $v \in \frac{n}{2}K_2$, then $d(v,a) = d(v,b) = 2$. So, $n(ab|\mathcal{C}(\mathrm{D}_n)) = 1$. We note that there exist $\binom{n-2}{2} = \frac{(n-2)(n-3)}{2}$ edges of the form ab where $a, b \in K_{n-2}$. So,

$$\sum_{\substack{e(=ab)\in\mathcal{C}(\mathrm{D}_n)\\ a,b\in K_{n-2}}} n(ab|\mathcal{C}(\mathrm{D}_n)) = \frac{(n-2)(n-3)}{2}. \tag{10}$$

Case 3: Let $e = ab$, where $a, b \in \frac{n}{2}K_2$. Thus, $a, b \in K_2$ for some n. Let v be a vertex of $\mathcal{C}(\mathrm{D}_n)$ such that $v \notin \{a, b\}$. Since $d(v,a) = d(v,b)$, so $n(ab|\mathcal{C}(\mathrm{D}_n)) = 1$.

We note that there exist $(n/2)$ edges of the form ab where $a, b \in \frac{n}{2}K_2$. So

$$\sum_{e(=ab)\in\mathcal{C}(\mathrm{D}_n),\quad a,b\in\frac{n}{2}K_2} n(ab|\mathcal{C}(\mathrm{D}_n)) = \frac{n}{2}. \tag{11}$$

Case 4: Let $e = ab$, where $a, b \in K_2$. Let v be a vertex of $\mathcal{C}(\mathrm{D}_n)$ such that $v \notin \{a, b\}$. Since $d(v,a) = d(v,b) = 1$, so $n(ab|\mathcal{C}(\mathrm{D}_n)) = 1$. Since there exists only 1 edge of the form ab where $a, b \in K_2$, so

$$\sum_{e(=ab)\in\mathcal{C}(\mathrm{D}_n),\quad a,b\in K_2} n(ab|\mathcal{C}(\mathrm{D}_n)) = 1. \tag{12}$$

Thus, using (8), (9), (10), (11) and (12) we obtain,

$$Sz(\mathcal{C}(\mathrm{D}_n)) = \sum_{e(=ab)\in\mathcal{C}(\mathrm{D}_n)} n(ab|\mathcal{C}(\mathrm{D}_n))$$

$$
\begin{aligned}
&= \sum_{e(=ab)\in\mathcal{C}(\mathrm{D}_n)} n_1(ab|\mathcal{C}(\mathrm{D}_n))n_2(ab|\mathcal{C}(\mathrm{D}_n))\\
&= \sum_{\substack{e(=ab)\in\mathcal{C}(\mathrm{D}_n)\\ a\in K_2 \text{ and } b\in K_{n-2}}} n(ab|\mathcal{C}(\mathrm{D}_n)) + \sum_{\substack{e(=ab)\in\mathcal{C}(\mathrm{D}_n)\\ a\in K_2 \text{ and } b\in \frac{n}{2}K_2}} n(ab|\mathcal{C}(\mathrm{D}_n))\\
&+ \sum_{\substack{e(=ab)\in\mathcal{C}(\mathrm{D}_n)\\ a,b\in K_{n-2}}} n(ab|\mathcal{C}(\mathrm{D}_n)) + \sum_{\substack{e(=ab)\in\mathcal{C}(\mathrm{D}_n)\\ a,b\in \frac{n}{2}K_2}} n(ab|\mathcal{C}(\mathrm{D}_n)) + \sum_{\substack{e(=ab)\in\mathcal{C}(\mathrm{D}_n)\\ a,b\in K_2}} n(ab|\mathcal{C}(\mathrm{D}_n))\\
&= 2(n-2)(n+1) + 2(2n^2-3n) + \frac{(n-2)(n-3)}{2} + \frac{n}{2} + 1\\
&= \frac{13n^2}{2} - 10n,
\end{aligned}
$$

which completes the proof. □

2.2. Szeged Index of Commuting Graph of Dicyclic group Dic_n

In this section, we now determine the Szeged index of commuting graph of the dicyclic group Dic_n.

Definition 1. The dicyclic group Dic_n having order $4n$ has the following representation:

$$\mathrm{Dic}_n = \langle a, x : a^{2n} = 1, a^n = x^2, ax = xa^{-1}\rangle.$$

Moreover, using [33, Corollary 2.6], we know that $\mathcal{C}(\mathrm{Dic}_n)$ has the following representation:

$$\mathcal{C}(\mathrm{Dic}_n) = K_2 \vee (K_{2n-2} \cup nK_2). \tag{13}$$

We now state without proof the following theorem as it can be proved by similar proof techniques as used in (2).

Theorem 3. *Given $n \in \mathbb{N}$, the Szeged index of $\mathcal{C}(\mathrm{Dic}_n)$ is given as follows:*

$$Sz(\mathcal{C}(\mathrm{Dic}_n)) = 2(13n^2 - 10n).$$

3. Energy of $\mathcal{C}(\mathrm{D}_n)$ and $\mathcal{C}(\mathrm{Dic}_n)$

Here, we determine the energy of the commuting graph of the dihedral and the dicyclic group. We prove that $\mathcal{C}(\mathrm{D}_n)$ and $\mathcal{C}(\mathrm{Dic}_n)$ are non-hyperenergetic.

Theorem 4. *If $n(\geq 3)$ is an odd number, then $\mathcal{C}(\mathrm{D}_n)$ is not hyperenergetic.*

Proof. Using [34, Proposition 5.1], the adjacency matrix of $\mathcal{C}(\mathrm{D}_n)$ has the following characteristic polynomial:

$$\Lambda(\mathcal{C}(\mathrm{D}_n); x) = x^{n-1}(x+1)^{n-2}\Big(x^3 + (-n+2)x^2 + (-2n+1)x + n^2 - 2n\Big). \tag{14}$$

Now, let us consider the polynomial $f(x) = x^3 + (2-n)x^2 + (1-2n)x + (n^2 - 2n)$.
Let the roots of $f(x)$ be $\lambda_1 \leq \lambda_2 \leq \lambda_3$ arranged in non-decreasing order.
Now, we have the following relations,

$$
\begin{aligned}
\lambda_1 + \lambda_2 + \lambda_3 &= n - 2 > 0,\\
\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 &= 1 - 2n,\\
\lambda_1\lambda_2\lambda_3 &= 2n - n^2 = n(2-n) < 0.
\end{aligned} \tag{15}
$$

Using (15), we find that the sum of roots of $f(x)$ is positive, while the product of roots of $f(x)$ is negative. Hence, we can assume that $f(x)$ has one negative root and 2 positive roots. Thus, we have $\lambda_1 \leq 0 \leq \lambda_2 \leq \lambda_3$.

Now,

$$f(0) = n^2 - 2n = n(n-2) > 0, \tag{16}$$

and

$$\begin{aligned} f(-n) &= -n^3 + (2-n)n^2 + (1-2n)(-n) + n^2 - 2n \\ &= -n(n-1)(2n-3) < 0. \end{aligned} \tag{17}$$

Since, $f(x)$ is a continuous function over the set of real numbers, using (17) and Bolzano's Intermediate Value Theorem, we find that $f(x) = 0$ has one real root in $(-n, 0)$. Hence, $-n < \lambda_1 < 0$.

Thus, we have

$$\begin{aligned} |\lambda_1| + |\lambda_2| + |\lambda_3| &= \lambda_1 + \lambda_2 + \lambda_3 - 2\lambda_1 \\ &= (n-2) - 2\lambda_1. \end{aligned} \tag{18}$$

Since $-n < \lambda_1 < 0$, we have,

$$\begin{aligned} & 2n > -2\lambda_1 > 0 \\ \implies & (n-2) + 2n > (n-2) - 2\lambda_1 > n-2 \\ \implies & 3n-2 > (n-2) - 2\lambda_1 > n-2. \end{aligned} \tag{19}$$

Using (18) and (19), we obtain

$$n - 2 < |\lambda_1| + |\lambda_2| + |\lambda_3| < 3n - 2. \tag{20}$$

Now, using (14), the energy of $\mathcal{C}(\mathrm{D}_n)$ is given as follows:

$$\mathcal{E}(\mathcal{C}(\mathrm{D}_n)) = (n-2) + |\lambda_1| + |\lambda_2| + |\lambda_3|.$$

Using (18) and (20), we have,

$$\begin{aligned} & (n-2) + (n-2) < \mathcal{E}(\mathcal{C}(\mathrm{D}_n)) < (3n-2) + (n-2) \\ \implies & 2(n-2) < \mathcal{E}(\mathcal{C}(\mathrm{D}_n)) < 4(n-1) = 4n - 4 < 4n - 2. \end{aligned}$$

Since $\mathcal{E}(\mathcal{C}(\mathrm{D}_n)) < 4n - 2$, we conclude that $\mathcal{C}(\mathrm{D}_n)$ is not hyperenergetic. Thus, the result follows. □

Theorem 5. *If $n (\geq 4)$ is an even number, then $\mathcal{C}(\mathrm{D}_n)$ is not hyperenergetic.*

Proof. Using [34, Proposition 5.9], the adjacency matrix of $\mathcal{C}(\mathrm{D}_n)$ has the following characteristic polynomial:

$$\Lambda(\mathcal{C}(\mathrm{D}_n); x) = (x+1)^{(\frac{n}{2}+n-2)}(x-1)^{(\frac{n}{2}-1)} \times \left(x^3 + (-n+1)x^2 + (-2n-1)x + 2n^2 - 5n - 1\right). \tag{21}$$

Now, let us consider the polynomial $f(x) = x^3 + (-n+1)x^2 + (-2n-1)x + 2n^2 - 5n - 1$.

Let the roots of $f(x)$ be $\lambda_1 \leq \lambda_2 \leq \lambda_3$ arranged in non-decreasing order.

Now, we have the following relations,

$$\begin{aligned} \lambda_1 + \lambda_2 + \lambda_3 &= n - 1 > 0, \\ \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 &= -(1+2n), \\ \lambda_1\lambda_2\lambda_3 &= -(2n^2 - 5n - 1). \end{aligned} \tag{22}$$

We note that since $n \geq 4$, $2n-5 \geq 3$. This implies that $n(2n-5) \geq 12$. Hence, $n(2n-5)-1 \geq 11 > 0$ for $n \geq 4$. Hence, $-(2n^2-5n-1) < 0$.

Using (22), we find that the sum of roots of $f(x)$ is positive, while the product of roots of $f(x)$ is negative. Hence, we can assume that $f(x)$ has one negative root and 2 positive roots. Thus, we have $\lambda_1 \leq 0 \leq \lambda_2 \leq \lambda_3$.

Now,

$$f(0) = 2n^2 - 5n - 1 > 0, \tag{23}$$

and

$$\begin{aligned} f\Big(-\frac{n}{2}\Big) &= -\Big(\frac{n}{2}\Big)^3 + (1-n)\Big(-\frac{n}{2}\Big)^2 + (-1-2n)\Big(-\frac{n}{2}\Big) + 2n^2 - 5n - 1 \\ &= -\frac{1}{8}\Big(n^3 - 10n^2 + 44n + 8\Big). \end{aligned} \tag{24}$$

Since $n \geq 4$, so $n^3 - 10n^2 + 44n + 8 \geq 0$, which in turn proves that $f\Big(-\frac{n}{2}\Big) < 0$.

Since, $f(x)$ is a continuous function over the set of real numbers, using (24) and Bolzano's Intermediate Value Theorem, we find that $f(x) = 0$ has one real root in $(-\frac{n}{2}, 0)$. Hence, $-\frac{n}{2} < \lambda_1 < 0$.

Thus, we have

$$\begin{aligned} |\lambda_1| + |\lambda_2| + |\lambda_3| &= \lambda_1 + \lambda_2 + \lambda_3 - 2\lambda_1 \\ &= (n-1) - 2\lambda_1. \end{aligned} \tag{25}$$

Since $-\frac{n}{2} < \lambda_1 < 0$, we have,

$$\begin{aligned} & n > -2\lambda_1 > 0 \\ \implies & (n-1) + n > (n-1) - 2\lambda_1 > n - 1 \\ \implies & 2n - 1 > (n-1) - 2\lambda_1 > n - 1. \end{aligned} \tag{26}$$

Using (**??**), we obtain

$$n - 1 < |\lambda_1| + |\lambda_2| + |\lambda_3| < 2n - 1. \tag{27}$$

Now, using (21), the energy of $\mathcal{C}(\mathrm{D}_n)$ is given as follows:

$$\begin{aligned} \mathcal{E}(\mathcal{C}(\mathrm{D}_n)) &= \Big(\frac{3n}{2} - 2\Big) + \Big(\frac{n}{2} - 1\Big) + |\lambda_1| + |\lambda_2| + |\lambda_3| \\ &= 2n - 3 + |\lambda_1| + |\lambda_2| + |\lambda_3|. \end{aligned}$$

Using (**??**), we obtain

$$\begin{aligned} & (2n-3) + (n-1) < \mathcal{E}(\mathcal{C}(\mathrm{D}_n)) < (2n-1) + (2n-3) \\ \implies & 3n - 4 < \mathcal{E}(\mathcal{C}(\mathrm{D}_n)) < 4n - 4. \end{aligned}$$

Since $\mathcal{E}(\mathcal{C}(\mathrm{D}_n)) < 4(n-1) < 4n-2$, we conclude that $\mathcal{C}(\mathrm{D}_n)$ is not hyperenergetic. □

Theorem 6. *If $n(\geq 4)$ is an even number, then $\mathcal{C}(\mathrm{Dic}_n)$ is not hyperenergetic.*

Proof. Using [34, Proposition 5.15], we find that the adjacency matrix of $\mathcal{C}(\mathrm{Dic}_n)$ has the following characteristic polynomial:

$$\Lambda(\mathcal{C}(\mathrm{Dic}_n); x) = (x+1)^{(3n-2)}(x-1)^{(n-1)} \times \left(x^3 + (-2n+1)x^2 + (-4n-1)x + 8n^2 - 10n - 1\right).$$

Consequently, the energy of $\mathcal{C}(\mathrm{Dic}_n)$ is given by:

$$\begin{aligned}\mathcal{E}(\mathcal{C}(\mathrm{Dic}_n)) &= 3n - 2 + n - 1 + |\lambda_1| + |\lambda_2| + |\lambda_3| \\ &= 4n - 3 + |\lambda_1| + |\lambda_2| + |\lambda_3|,\end{aligned}$$

where $\lambda_1, \lambda_2, \lambda_3$ are roots of the equation $x^3 + (-2n+1)x^2 + (-4n-1)x + 8n^2 - 10n - 1 = 0$. Using similar proof techniques as used in (5), the result follows. □

4. Conclusion

The study of the *commuting graph* of a finite group is motivated by its ability to provide a visual and structural perspective on the group's internal relationships, such as centralizers, conjugacy classes, and the center of the group. It helps in classifying groups (abelian, nilpotent, solvable), investigating central and non-central structure, studying automorphisms and group actions, identifying normal subgroups, and providing computational tools for group analysis.

In this paper, we compute the Szeged index of the commuting graph of the dihedral group and the dicyclic group. We also calculate the energy of the commuting graphs of the dihedral group and the dicyclic group, and prove that they are always nonhyperenergetic. Though the Szeged index of a graph is an important topological index, it has not been calculated explicitly for different algebraic graphs yet. We invite the readers to calculate the Szeged index of various other graphs existing in the literature, namely the zero divisor graph, comaximal graph, power graph, and so on.

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ORCID

Subarsha Banerjee https://orcid.org/0000-0002-7029-7650

Ankita Agarwal https://orcid.org/0009-0004-5664-9252

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