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Research Paper

Numerical Solution of Time Fractional Klein-Gordon Equation in Framework of the Yang-Abdel-Cattani Fractional Derivative Operator

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Abstract

This paper investigates the analytical results of both linear and nonlinear time-fractional Klein-Gordon equations using the novel Yang-Abdel-Cattani derivative operator. In order to solve the proposed time-fractional Klein-Gordon equations, we applied the Laplace Adomian decomposition technique. The effectiveness of this operator is demonstrated through three test problems. These include both linear and nonlinear time-fractional Klein-Gordon equations. Furthermore, the influence of different fractional Brownian motion values on the solution profiles is analyzed and presented graphically.

Key Words: Klein-Gordon Equations, Yang Abdel Cattani Fractional Derivative Operator, Existence and Uniqueness

AMS 2020 Classification: 34A08, 34A12, 34A45

1. Introduction

Fractional calculus generalizes traditional integration and differentiation to non-integer orders. This field has intrigued scholars for over three centuries. The non-local nature of fractional differential equations makes them particularly adept at modeling a wide range of phenomena in both scientific and engineering contexts. This growing academic interest has not only resolved practical issues but also enabled precise characterization of nonlinear phenomena, especially in fluid mechanics, where traditional continuum assumptions often fall short. As a result, fractional models have become prime candidates for such applications[1, 2, 3]. Moreover, fractional derivatives are powerful mathematical tools for representing processes with memory and hereditary properties[4, 5]. Researchers have developed various methods to solve both linear and nonlinear fractional differential equations, demonstrating the advantages and effectiveness of fractional calculus in these areas[6, 7, 8].

In 1927, the Klein-Gordon equation was introduced by Oscar Klein and Walter Gordon as a fascinating differential equation. In his search for an equation defining de Broglie waves, Schrodinger initially considered the Klein-Gordon equation as a quantum wave equation. Solid-state physics, elementary particle behavior, dispersive wave phenomena, dislocation propagation in crystals, nonlinear optics, relativistic physics, and quantum field theory are several scientific applications that rely on the Klein-Gordon equation [9, 10]. The authors of [11] studied nonlinear Schrodinger equations of fractional order. In[12], authors have a study of implicit-impulsive differential equations involving the Caputo-Fabrizio fractional derivative. Ullah et al. [13] have proposed advancing non-linear PDE solutions: the modified Yang transform method with the Caputo-Fabrizio fractional operator. Usta et al.[14] have studied analytical solutions within local fractional Volterra and Abel's integral equations via the Yang-Laplace transform. In [15], authors have investigated the existence of solutions to integral equations in the form of quadratic Urysohn type with Hadamard fractional variable order integral operator. The authors of [16] studied a generalized Mittag-Leffler function and a modified general class of functions which is reducible to several special functions. Golmankhaneh et al. [17] have investigated the fractional order Klein-Gordon equations (KGEs) using the homotopy perturbation method (HPM). Tamsir et al. [18] have solved the fractional order KGEs using the fractional reduced differential transform method. Authors in [19] have investigated the numerical computation of Klein-Gordon equations arising in quantum field theory by using the homotopy analysis transform method. A number of articles in [18, 20, 21] provide recent publications on fractional Klein-Gordon equations.

In this study, we apply the Laplace Adomian Decomposition Method (LADM) to the time fractional order Klein-Gordon equation, yielding results in the YAC sense. The computational process of this method is straightforward, and graphical representations substantiate our findings. Our primary focus is on the fractional model of the Klein-Gordon equation[18].

$$^{YAC}\mathscr{D}_t^q\phi(r,t) = \phi_{rr}(r,t) + u\phi(r,t) + v\phi^2(r,t) + w\phi^3(r,t), \quad t > 0 \tag{1}$$

Subject to the initial conditions

$$\phi(r,0) = \phi_0, \;\; r \in R$$

Where $^{YAC}\mathscr{D}_t^q\phi = \dfrac{\partial^q\phi}{\partial t^q}$, $\phi_{rr} = \dfrac{\partial^2\phi}{\partial r^2}$ and u, v, w are real constants. Yang et al. [22] introduced a generalized fractional derivative by employing the Rabotnov exponential function as a non-singular kernel. Notably, when $q = 1$, Eq (1) reduces to the classical Klein-Gordon equations (KGEs), which have numerous applications in areas such as solid-state physics, nonlinear optics, and quantum field theory [23]. The recurrence of initial states and soliton interactions in collisionless plasma are just a few examples of the equation's many applications. As a fundamental equation in mathematical physics, it has been widely researched in relation to solitons and condensed matter physics [24, 25, 26]. Various methods have been applied to solve the KGEs, including the variational iteration method (VIM), homotopy perturbation method (HPM), fractional reduced differential transform method (FRDTM), and Adomian decomposition method (ADM) [18, 25, 26].

The structure of this work is organized as follows: This section provides an introduction to the historical background of fractional calculus, the Klein-Gordon equations, and non-integer derivatives. Section 2 outlines the basic definitions and properties of the newly proposed arbitrary-order YAC derivative, along with an analysis of its essential features. In Section 3, the Laplace Adomian decomposition method is discussed, as well as the Laplace integral transform of new derivative properties through a theorem. Section 4 addresses the existence and uniqueness of solutions. Section 5 presents three numerical examples and includes graphs to illustrate the results. Finally, concluding remarks are provided in Section 6.

2. The Yang-Abdel-Cattani Fractional Calculus

This section provides a brief overview of the arbitrary-order YAC derivative, integral operators, and fractional Rabotnov exponential functions.

Definition 1. [22, 27] The Rabotnov exponential function of non integer order k is defined below

$$\varrho_q(\iota u^q) = \sum_{k=0}^{\infty} \frac{\iota^k u^{[(k+1)(q+1)]-1}}{\Gamma[(k+1)(q+1)]}, \;\; u \in C, \; k, q \in \Re > 0.$$

and its Laplace transform is:

$$\mathcal{L}\left\{\varrho_q(\iota u^q);\xi\right\} = \frac{1}{\xi^{q+1}}\frac{1}{1-\eta\xi^{-(q+1)}}, \;\; |(\eta\xi^{-(q+1)})| < 1.$$

Definition 2. [22, 27] The generalized YAC derivative of arbitrary order can be expressed using the fractional Rabotnov exponential function (FREF) as follows:

$$^{YAC}\mathscr{D}_t^{mq}\phi(r,t) = \int_0^t \varrho_q[-\eta(t-\chi)^q]\phi^m(r,\chi)d\chi, \ \ m \in N$$

where ϱ_q represents the Rabotnov exponential function of order q, and its Laplace transform is characterized as

$$\mathcal{L}\left\{^{YAC}\mathscr{D}_t^{mq}\phi(r,t)\right\} = \frac{1}{\xi^{q+1}}\frac{1}{1+\eta\xi^{-(q+1)}}[\xi^m\phi(r,\chi) - \sum_{p=1}^{m}\xi^{m-p}\phi^p(r,0)]$$

Definition 3. [22, 27] The associated integral of order q, with $0 < \eta \in \mathbb{R}$ and $q \in (0,1]$, is defined as follows:

$$I_0^{YAC}\phi(t) = \int_0^t \varrho_q[-\eta(t-\chi)^q]\phi(\chi)d\chi$$

[22] and its $\mathcal{L}-$transform is

$$\mathcal{L}\left\{I_0^{YAC}\phi(t):\xi\right\} = \frac{1}{\xi^{q+1}}\frac{L[\phi(t)]}{1-\eta\xi^{-(q+1)}}$$

Definition 4. [27] The Prabhakar function is defined as follows:

$$\Psi^{\mu}_{q,\rho}\nu = \sum_{k=0}^{\infty}\frac{(\mu)_k}{k!\Gamma(qk+\rho)}\nu^k, \ \ Re(q)>0, \ Re(\rho)>0, \ \mu>0$$

Where $(\mu)_k$ is a Pochhammer notation [28] and has the following form

$$(\mu)_k = \begin{cases} \mu(\mu+1)(\mu+2)...(\mu+k-1) = \dfrac{\Gamma(\mu+k)}{\Gamma(\mu)}, & k \in Z^+, \\ 1, \ k=0 \\ \mu(\mu-1)(\mu-2)...(\mu-k+1) = \dfrac{\Gamma(\mu+1)}{\Gamma(\mu-k+1)}, & k \in Z^- \end{cases}$$

and its Laplace integral transform has the form below:

$$\mathcal{L}\left\{\nu^{\rho-1}\Psi^{\mu}_{q,\rho}(\eta\nu^q)\right\} = \xi^{-\rho}(1-\eta\xi^{-q})^{-\mu}, \ \ |\eta\xi^{-q}| < 1.$$

The Prabhakar function is connected with the following special functions. (see [28]).

$$\Psi^{-\mu}_{q,\rho+1}\nu = \frac{\Gamma(\mu+1)}{\Gamma(\mu q+\rho+1)}E^{\rho}_{\mu}(\nu,q), \tag{2}$$

With $E^{\rho}_{\mu}(\nu,q)$ is a polynomial of degree μ in ν^q studed in [28].

$$\Psi^{-\mu}_{1,\rho+1}\nu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\rho+1)}L^{\rho}_{\mu}. \tag{3}$$

where L^{ρ}_{μ} represents the Laguerre polynomial. In this study, we will utilize the Prabhakar functions, defined in Equations (2) and (3), to derive the results for an arbitrary-order Klein-Gordon equation within the context of the YAC derivative.

Lemma 1. [29] Let $q,\rho > 0, \eta \in \Re$ and μ be a positive integer. Then

$$L\left\{t^{\rho-1}\Psi^{-\mu}_{q,\rho}(\eta\nu^q)\right\}(\xi) = \xi^{-\rho}(1-\eta\xi^{-q})^{\mu}, \ \ \xi > 0.$$

3. Laplace Adomian Decomposition Method

Consider the following partial differential equation of arbitrary order, framed within the YAC derivative context:

$$^{YAC}\mathscr{D}_t^q \phi(r,t) = \zeta_1 \phi(r,t) + H_1(r,t). \tag{4}$$

The linear and nonlinear components of the first equation in system (1) are denoted by ζ_1 and H_1, respectively, with the initial condition being:

$$\phi(r,0) = h(r,t).$$

Next, applying the Laplace transform to Equation (4) on both sides results in:

$$\frac{1}{\xi^{q+1}} \frac{\xi \mathcal{L}\left[\phi(r,t)\right] - \phi(r,0)}{1+\eta\xi^{-q-1}} = \mathcal{L}\left[\zeta_1\phi(r,t) + H_1\phi(r,t)\right].$$

$$\mathcal{L}[\phi(r,t)] = \frac{\phi(r,0)}{\xi} + \xi^q(1+\eta\xi^{-q-1})\mathcal{L}[\zeta_1\phi(r,t) + H_1\phi(r,t)]. \tag{5}$$

The solution is:

$$\zeta_1[\phi(r,t)] = \sum_{m=0}^{\infty} \phi_m(r,t) \tag{6}$$

The nonlinear term in the problem is described as:

$$H_1[\phi(r,t)] = \sum_{m=0}^{\infty} K_m. \tag{7}$$

where

$$K_m = \frac{1}{m!}\left[\frac{d^m}{dp^m}(H_1 \sum_{m=0}^{\infty} p^m \phi_m)\right]_{p=0}$$

are called Adomian polynomials. Substituting Equations (6) and (7) into Equation (5) gives:

$$\mathcal{L}[\sum_{m=0}^{\infty}\phi_m(r,t)] = \frac{\phi(r,0)}{\xi} + \xi^q(1+\eta\xi^{-q-1})\mathcal{L}\left[\sum_{m=0}^{\infty}\phi_m(r,t) + \sum_{m=0}^{\infty} K_m\right]$$

Using the decomposition method, we derive:

$$\mathcal{L}[\phi_0(r,t)] = \frac{\phi(r,0)}{\xi}$$

$$\mathcal{L}[\phi_{m+1}(r,t)] = \xi^q(1+\eta\xi^{-q-1})\mathcal{L}[\zeta_1\phi_m(r,t) + H_1\phi_m(r,t)]$$

Theorem 1. *The Laplace transform of the fractional Rabotnov exponential kernel is defined as follows*

$$\mathcal{L}\left\{^{YAC}\mathscr{D}_t^{mq}\phi(r,t)\right\} = \frac{1}{\xi^{q+1}}\frac{1}{1+\eta\xi^{-(q+1)}}\left[\xi^m\phi(r,\chi) - \sum_{p=1}^{m}\xi^{m-p}\phi^p(r,0)\right]$$

Theorem 2. *Let $\phi \in H^1(c,d), d > c, q \in (0,1]$. Then, $^{YAC}\mathscr{D}_t^{mq}$ is the m^{th}-order YAC fractional derivative operator. Then, the Laplace transform of $^{YAC}\mathscr{D}_t^{mq}$ is*

$$\mathcal{L}\left\{^{YAC}\mathscr{D}_t^{mq}\phi(r,t)\right\} = \frac{1}{\xi^{q+1}}\frac{1}{1+\eta\xi^{-(q+1)}}\left[\xi^m\phi(r,\chi) - \sum_{p=1}^{m}\xi^{m-p}\phi^p(r,0)\right]$$

Proof. The YAC fractional derivative with a Rabotnov kernel is defined as:

$$^{YAC}\mathscr{D}_t^q\phi(r,t) = \int_0^t \varrho_q[-\eta(t-\chi)^q]\phi'(r,\chi)d\chi.$$

By the definition of LT

$$\mathcal{L}\left\{^{YAC}\mathscr{D}_t^q\phi(r,t)\right\} = \int_0^\infty e^{-\xi t}\left\{\int_0^t \varrho_q[-\eta(t-\chi)^q]\phi'(r,\chi)d\chi\right\}dt$$

By applying the convolution theorem, we obtain:

$$\mathcal{L}\left\{^{YAC}\mathscr{D}_t^q\phi(r,t)\right\} = \mathcal{L}\left\{\phi'(r,t)\right\} * \mathcal{L}\left\{\varrho_q(-\eta t^q)\right\}$$

we obtained

$$\mathcal{L}\left\{^{YAC}\mathscr{D}_t^q\phi(r,t)\right\} = \frac{1}{\xi^{q+1}}\frac{1}{1+\eta\xi^{-(q+1)}}\left[\xi\mathcal{L}[\phi(r,\chi)] - \phi(r,0)\right]$$

Similarly,

$$\mathcal{L}\left\{^{YAC}\mathscr{D}_t^{2q}\phi(r,t)\right\} = \frac{1}{\xi^{q+1}}\frac{1}{1+\eta\xi^{-(q+1)}}\left[\xi^2 L[\phi(r,\chi)] - \xi\phi'(r,0) - \phi(r,0)\right]$$

Through mathematical induction, we derive:

$$\mathcal{L}\left\{^{YAC}\mathscr{D}_t^{mq}\phi(r,t)\right\} = \frac{1}{\xi^{q+1}}\frac{1}{1+\eta\xi^{-(q+1)}}\left[\xi^m\phi(r,\chi) - \sum_{p=1}^{m}\xi^{m-p}\phi^p(r,0)\right]$$

which completes the proof. □

4. Existence and Uniqueness

Theorem 3. *The YAC derivative operator of order q satisfies the Lipschitz condition, where a is the Lipschitz constant. Specifically, this can be expressed as:*

$$\left\|^{YAC}\mathscr{D}_t^q\phi_1(r,t) -^{YAC}\mathscr{D}_t^q\phi_2(r,t)\right|| \leq a||\phi_1(r,t) - \phi_2(r,t)\right\|$$

Proof. By utilizing the YAC fractional derivative of order q, the following result is obtained:

$$\begin{aligned}\left\|^{YAC}\mathscr{D}_t^q\phi_1(r,t) -^{YAC}\mathscr{D}_t^q\phi_2(r,t)\right\| &= \left\|\int_0^t \varrho_q[-\eta(t-\chi)^q]\phi_1'(r,\chi)d\chi - \int_0^t \varrho_q[-\eta(t-\chi)^q]\phi_2'(r,\chi)d\chi\right\| \\ &= \left\|\int_0^t \varrho_q[-\eta(t-\chi)^q][\phi_1'(r,\chi) - \phi_2'(r,\chi)]d\chi\right\|\end{aligned}$$

$$\begin{aligned}\left\|^{YAC}\mathscr{D}_t^q\phi_1(r,t) -^{YAC}\mathscr{D}_t^q\phi_2(r,t)\right\| &\leq \int_0^t \|\varrho_q[-\eta(t-\chi)^q]\|\,\|[\phi_1'(r,\chi) - \phi_2'(r,\chi)]d\chi\| \\ &\leq a\,\|[\phi_1(r,\chi) - \phi_2(r,\chi)]\|\end{aligned}$$

Finally, we get the following result:

$$\left\|^{YAC}\mathscr{D}_t^q\phi_1(r,t) -^{YAC}\mathscr{D}_t^q\phi_2(r,t)\right\| \leq a\,\|\phi_1(r,t) - \phi_2(r,t)\|$$

which completes the proof. □

Theorem 4. *Let us assume that the function* $f(r,t,\phi,\phi_r,\phi_{rr})$ *satisfies the Lipschitz condition as*

$$|f(r,t,\phi,\phi_r,\phi_{rr}) - f(r,t,\phi_1,\phi_{1r},\phi_{1rr})| \leq K|\phi-\phi_1| + M|\phi_r-\phi_{1r}| + N|\phi_{rr}-\phi_{1rr}|$$

We also assume that

$$|\phi_r - \phi_{1r}| \leq \alpha|\phi-\phi_1|$$
$$|\phi_{rr} - \phi_{1rr}| \leq \beta|\phi-\phi_1|$$

where α *and* β *are elements of* $\mathbb{R}^+$*, there exists a unique solution for the following time fractional differential equation:*

$$^{YAC}\mathscr{D}_t^q\phi = \phi_{rr} + u\phi + v\phi^2 + w\phi^3,$$

Proof. We define

$$\varpi(\phi,r) = f(r,t,\phi,\phi_r,\phi_{rr}) = \phi_{rr} + u\phi + v\phi^2 + w\phi^3$$

We first show that $\varpi(\phi,r)$ satisfies Lipschitz condition. Consider

$$\begin{aligned}
||\varpi(\phi,r) - \varpi(\phi_1,r)|| &= ||f(r,t,\phi,\phi_r,\phi_{rr}) - f(r,t,\phi_1,\phi_{1r},\phi_{1rr})|| \\
&\leq K||\phi-\phi_1|| + M||\phi_r-\phi_{1r}|| + N||\phi_{rr}-\phi_{1rr}|| \\
&\leq [K + M\alpha + N\beta]||\phi-\phi_1|| \\
&= A||\phi-\phi_1||
\end{aligned}$$

where $A = K + M\alpha + N\beta \in \Re^+$.

Using Picard's theorem, we obtain

$$\phi(r,t) = \phi(r,0) + \int_0^t \varrho_q[-\eta(t-\chi)^q]\varpi(\phi,r(\chi))d\chi.$$

For convenience, we write

$$\int_0^t \varrho_q[-\eta(t-\chi)^q]\varpi(\phi,r(chi))d\chi = I_{YAC}^q\varpi(\phi,r(\chi))d\chi.$$

Finally, we have

$$\begin{aligned}
\|\phi(r,t) - \phi(r,0)\| &= \|I_{YAC}^q\varpi(\phi,r(\chi))d\chi\|, \\
&= \left\|\int_0^t \varrho_q[-\eta(t-\chi)^q]\varpi(\phi,r(\chi))d\chi\right\| \\
&\leq \int_0^t \|\varrho_q[-\eta(t-\chi)^q]\varpi(\phi,r(\chi))\|\, d\chi \\
&\leq AI_{YAC}^q(1)
\end{aligned}$$

Now, we consider

$$\begin{aligned}
||\phi(r,t) - \phi(r,0)|| &= ||I_{YAC}^q\varpi(\phi,r,\chi)d\chi - I_{YAC}^q\varpi(\phi_1,r(\chi))d\chi||, \\
&\leq I_{YAC}^q||\varpi(\phi,r,\chi)d\chi - \varpi(\phi_1,r(\chi))d\chi||, \\
&\leq AI_{YAC}^q||\phi-\phi_1||
\end{aligned}$$

For the above mapping to be a contraction, the condition must be met as follows:

$$AI_{YAC}^q \leq 1$$

$$I_{YAC}^q \leq \frac{1}{A}$$

Thus, the existence and uniqueness of the solution can be established as a direct consequence of the Banach fixed point theorem. □

5. Solutions of Time Fractional Klein-Gordon Equations

In this section, we will explore the applications of the newly introduced Yang-Abel-Cattani (YAC) fractional derivative by examining three numerical examples involving both linear and nonlinear time fractional Klein-Gordon equations.

Example 1. Consider the linear time-fractional order Klein-Gordon equation [18]:

$$^{YAC}\mathscr{D}_t^q\phi = \phi_{rr} + \phi, \quad t \geq 0, \tag{8}$$

subject to the initial condition

$$\phi(r,0) = 1 + sin(r)$$

When the Laplace transform is applied to Equation (8) on both sides, we have:

$$\frac{1}{\xi^{q+1}}\frac{\xi\mathcal{L}[\phi(r,t)] - \phi(r,0)}{1+\eta\xi^{-q-1}} = L[\phi_{rr} + \phi]$$

$$\mathcal{L}[\phi(r,t)] = \frac{\phi(r,0)}{\xi} + \xi^q(1+\eta\xi^{-q-1})\mathcal{L}[\phi_{rr} + \phi] \tag{9}$$

Taking the inverse Laplace transform on both sides of Equation (9) yields:

$$\phi(r,t) = \phi(r,0) + \mathcal{L}^{-1}\left\{\xi^q(1+\eta\xi^{-q-1})\mathcal{L}[\phi_{rr} + \phi]\right\}$$

Following the Adomian decomposition method, the procedure results in:

$$\sum_{m=0}^{\infty}\phi_m(r,t) = \phi(r,0) + \mathcal{L}^{-1}\left\{\xi^q(1+\eta\xi^{-q-1})\mathcal{L}[\sum_{m=0}^{\infty}\phi_m(r,t)]\right\}$$

Now we will find the values of $\phi_1(r,t)$,$\phi_2(r,t)$,...,$\phi_m(r,t)$ by putting $m = 1, 2, 3, ...$. Estimation of the first iteration $\phi_1(r,t)$: setting $m = 1$ we obtain

$$\phi_1(r,t) = \mathcal{L}^{-1}\left\{\xi^q(1+\eta\xi^{-q-1})L[\phi_{0rr} + \phi_0]\right\} \tag{10}$$

Now using the value of starting guess $\phi_0(r,t) = 1 + sin(r)$ in above equation (10) we obtain:

$$\phi_1(r,t) = t^{-q}\Psi^{-1}_{q+1,-q+1}(-\eta t^{q+1})$$

The next few terms are expressed as:

$$\phi_2(r,t) = t^{-2q}\Psi^{-2}_{q+1,-2q+1}(-\eta t^{q+1})$$

$$\phi_3(r,t) = t^{-3q}\Psi^{-3}_{q+1,-3q+1}(-\eta t^{q+1})$$

$$\phi_m(r,t) = t^{-mq}\Psi^{-m}_{q+1,-mq+1}(-\eta t^{q+1})$$

Therefore the final solution of (8) is

$$\phi(r,t) = 1 + sin(r) + \sum_{m=1}^{\infty} t^{-qm}\Psi^{-m}_{q+1,-mq+1}(-\eta t^{q+1})$$

An exact solution to the classical Klein-Gordon equation, specifically equation (8), can be derived. This solution, which aligns with the results previously obtained by M. Tamsir et al. [18] using the fractional reduced differential transform method, offers a consistent analytical approach.

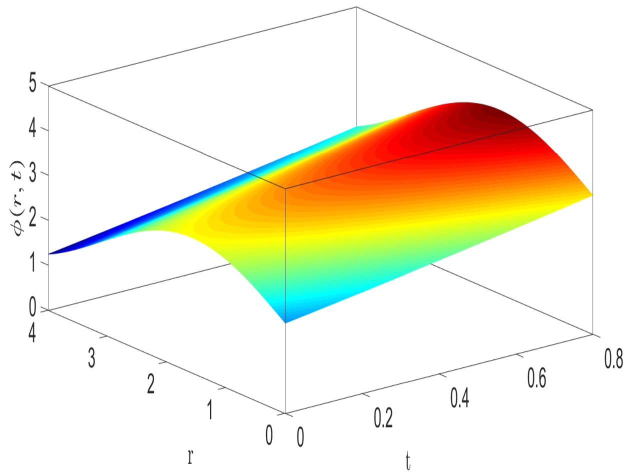

Fig. 1. 3D view of example 1 with the parameters $q = 1$ and $\eta = 0.5$.

Example 2. Consider the nonlinear time fractional order Klein-Gordon equation [18]

$$^{YAC}\mathscr{D}_t^q \phi = \phi_{rr} - \phi^2, \quad t > 0, \tag{11}$$

subject to the initial condition as follows

$$\phi(r,0) = 1 + sin(r)$$

Operating the Laplace transform on both sides of Equation (11) leads to:

$$\frac{1}{\xi^{q+1}} \frac{\xi \mathcal{L}[\phi(r,t)] - \phi(r,0)}{1 + \eta \xi^{-q-1}} = \mathcal{L}[\phi_{rr} - \phi^2]$$

$$\mathcal{L}[\phi(r,t)] = \frac{\phi(r,0)}{\xi} + \xi^q (1 + \eta \xi^{-q-1}) \mathcal{L}[\phi_{rr} - \phi^2] \tag{12}$$

Implementing the inverse Laplace transform on left and right sides of Equation (12) results in:

$$\phi(r,t) = \phi(r,0) + \mathcal{L}^{-1} \left\{ \xi^q (1 + \eta \xi^{-q-1}) \mathcal{L}[\phi_{rr} - \phi^2] \right\}$$

Through the Adomian decomposition method, the solution is:

$$\sum_{m=0}^{\infty} \phi_m(r,t) = \phi(r,0) + \mathcal{L}^{-1} \left\{ \xi^q (1 + \eta \xi^{-q-1}) \mathcal{L}[\sum_{m=0}^{\infty} \phi_m(r,t) - \sum_{m=0}^{\infty} K_m(r,t)] \right\}$$

where Adomian polynomial components $K_m(r,t)$ are given as follows:

$$K_0 = \phi_0^2; \;\; K_1 = 2\phi_0\phi_1; \;\; K_2 = 2\phi_0\phi_1 + \phi_1^2$$

and so on. For $m = 0, 1, 2, ...$

$$\phi_1(r,t) = \mathcal{L}^{-1} \left\{ \xi^q (1 + \eta \xi^{-q-1}) \mathcal{L}[\phi_{0rr} - \phi_0^2] \right\} \tag{13}$$

Now, putting the value of starting guess $\phi_0(r,t) = 1 + sin(r)$ in Equation (13), we obtain:

$$\phi_1(r,t) = -[1 + 3sin(r) + sin^2(r)] t^{-q} \Psi_{q+1,-q+1}^{-1}(-\eta t^{q+1})$$

Subsequently, the following terms can be expressed as:

$$\phi_2(r,t) = [11sin(r) + 12sin^2(r) + 2sin^3(r)]t^{-2q}\Psi^{-2}_{q+1,-2q+1}(-\eta t^{q+1})$$

$$\phi_3(r,t) = [18 - 57sin(r) - 160sin^2(r) - 82sin^3(r) - 10sin(4r)]t^{-3q}\Psi^{-3}_{q+1,-3q+1}(-\eta t^{q+1})$$

$$...$$

Therefore the general solution of (11) is

$$\phi(r,t) = 1 + sin(r) - [1 + 3sin(r) + sin^2(r)]t^{-q}\Psi^{-1}_{q+1,-q+1}(-\eta t^{q+1}) + [11sin(r) + 12sin^2(r) + 2sin^3(r)]t^{-2q}\Psi^{-2}_{q+1,-2q+1}(-\eta t^{q+1}) + [18 - 57sin(r) - 160sin^2(r) - 82sin^3(r) - 10sin(4r)]t^{-3q}\Psi^{-3}_{q+1,-3q+1}(-\eta t^{q+1}) - ... \quad (14)$$

Eq (14) is the approximate solution for the nonlinear time fractional Klein-Gordon equation. The same solution was obtained by M.Tamsir et al [18] using the fractional reduced differential transform method.

t	r	FRDTM [18]	Our Method
	-2	0.0925	0.0907
	-1.5	0.0045	0.0025
	-1	0.1602	0.1585
	-0.5	0.5210	0.5206
	0	0.9980	1.0000
0.002	0.5	1.4741	1.4794
	1	1.8330	1.8415
	1.5	1.9876	1.9975
	2	1.9002	1.9093
	2.5	1.5922	1.5985
	3	1.1382	1.1411

Table 1. Comparison study between the fractional reduced differential transform method and our method for numerical Example 2, when $q = 1$ and $\eta = 1E - 10$

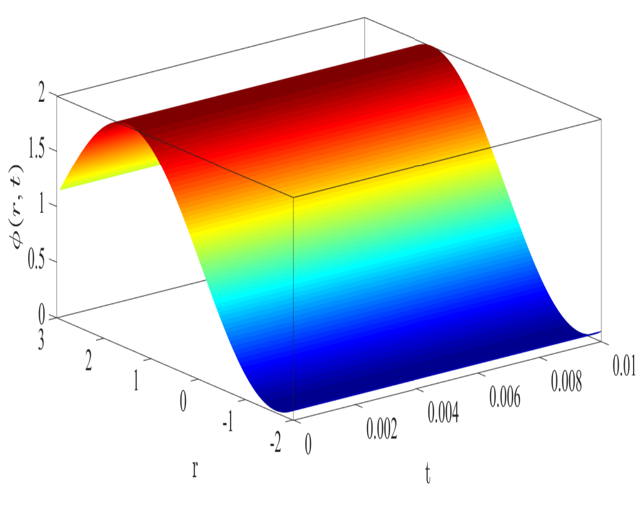

Fig. 2. 3D view of example 2 with the parameters $q = 1$ and $\eta = 1E - 10$.

Example 3. Consider the nonlinear time fractional order Klein-Gordon equation [18]

$$ {}^{YAC}\mathscr{D}_t^q \phi = \phi_{rr} - \phi + \phi^3, \quad t > 0, \tag{15} $$

subject to the initial condition as follows

$$ \phi(r,0) = -sech(r). $$

Applying the Laplace transform to Equation (15) on two sides provides:

$$ \frac{1}{\xi^{q+1}} \frac{\xi \mathcal{L}[\phi(r,t)] - \phi(r,0)}{1 + \eta \xi^{-q-1}} = \mathcal{L}[(\phi_{rr} - \phi) + \phi^3] $$

$$ \mathcal{L}[\phi(r,t)] = \frac{\phi(r,0)}{\xi} + \xi^q (1 + \eta \xi^{-q-1}) \mathcal{L}[(\phi_{rr} - \phi) + \phi^3] \tag{16} $$

After taking the inverse Laplace transform on two sides of Equation (16), we get:

$$ \phi(r,t) = \phi(r,0) + \mathcal{L}^{-1}\left\{ \xi^q (1 + \eta \xi^{-q-1}) \mathcal{L}[(\phi_{rr} - \phi) + \phi^3] \right\} $$

By following the Adomian decomposition method, the next terms are derived as:

$$ \sum_{m=0}^{\infty} \phi_m(r,t) = \phi(r,0) + \mathcal{L}^{-1}\left\{ \xi^q (1 + \eta \xi^{-q-1}) \mathcal{L}\left[\sum_{m=0}^{\infty} \phi_m(r,t) + \sum_{m=0}^{\infty} K_m(r,t) \right] \right\} $$

where Adomian polynomial components $K_m(r,t)$ are given as follows:

$$ K_0 = \phi_0^3; \;\; K_1 = \phi_0 \phi_1^2 + 2\phi_1 \phi_0^2; \;\; K_2 = 3\phi_0^2 \phi_2 + 3\phi_0 \phi_1^2 $$

and so on. For $m = 0, 1, 2, ...$

$$ \phi_1(r,t) = \mathcal{L}^{-1}\left\{ \xi^q (1 + \eta \xi^{-q-1}) \mathcal{L}\left[\phi_{0rr} - \phi_0 + \phi_0^3 \right] \right\} \tag{17} $$

Now, putting the value of $\phi(r,t)$ and using the value of initial guess $\phi_0(r,t) = -sech(r)$ in equation (17), we get

$$ \phi_1(r,t) = -[2sech(r) - 3sech^3(r)] t^{-q} \Psi_{q+1,-q+1}^{-1}(-\eta t^{q+1}) $$

Consequently, the next few component have the following expressions:

$$ \phi_2(r,t) = -[3sech(r) - 34sech^3(r) - 18sech^5(r)] t^{-2q} \Psi_{q+1,-2q+1}^{-2}(-\eta t^{q+1}) $$

$$ \phi_3(r,t) = -[64sech^3(r) - 288sech^5(r) + 240sech^7(r)] t^{-3q} \Psi_{q+1,-3q+1}^{-3}(-\eta t^{q+1}) $$

$$ ... $$

Therefore the general solution of (15) is

$$ \begin{aligned} \phi(r,t) = -sech(r) - [2sech(r) - 3sech^3(r)] t^{-q} \Psi_{q+1,-q+1}^{-1}(-\eta t^{q+1}) - [3sech(r) - 34sech^3(r) \\ -18sech^5(r)] t^{-2q} \Psi_{q+1,-2q+1}^{-2}(-\eta t^{q+1}) - [64sech^3(r) - 288sech^5(r) \\ + 240sech^7(r)] t^{-3q} \Psi_{q+1,-3q+1}^{-3}(-\eta t^{q+1}) - ... \end{aligned} \tag{18} $$

Eq (18) is the approximate solution for the nonlinear time fractional Klein-Gordon equation, clearly in complete agreement. with the results given by M. Tamsir et al [18] using FRDTM.

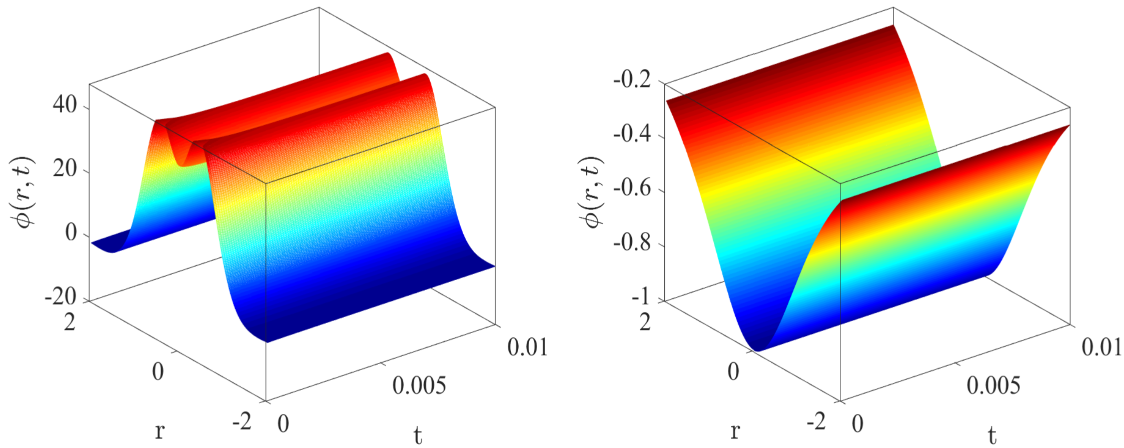

Fig. 3. Three dimensional plot of example 3 with the parameters; a. $q = 0.01$ and $\eta = 1$ and b. $q = 1$ and $\eta = 1E - 10$

t	r	FRDTM [18]	Our Method
	-2	-0.2682	-0.2658
	-1.5	-0.4282	-0.4251
	-1	-0.6503	-0.6481
	-0.5	-0.8848	-0.8868
0.005	0	-0.9944	-1.0000
	0.5	-0.8848	-0.8868
	1	-0.6503	-0.6481
	1.5	-0.4282	-0.4251
	2	-0.2682	-0.2658

Table 2. Comparison study between the fractional reduced differential transform method and our method for numerical Example 3, when $q = 1$ and $\eta = 1E - 10$

6. Conclusion

This work aimed to evaluate the YAC fractional derivative operator's effectiveness in solving linear and nonlinear time-fractional Klein-Gordon equations analytically. This study revealed several noteworthy properties of the YAC fractional derivative operator. To evaluate the accuracy and effectiveness of the approach and support the underlying theoretical concepts, three computational examples were carried out. These examples demonstrate that as fractional Brownian motion approaches non-fractional Brownian motion, the solution profile demonstrates a decay trend. The solutions derived using the YAC fractional derivative operator align closely with those reported by M. Tamsir et al. [18].

Declarations

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