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Research Paper

A Unified Explicit Binet Formula for 3^{rd}-Order Linear Recurrence Relations

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Abstract

In this paper, third order generalized linear recurrence relation $V_n\left(a_j, p_j\right) = p_1 V_{n-1} + p_2 V_{n-2} + p_3 V_{n-3}$, $p_3 \neq 0$, is studied to generate a generalized Tribonacci sequence, where $p_j, V_j = a_j$ are arbitrary integers. Generalized generating function for the 3^{rd} order general tribonacci sequence is derived, and then new unified explicit generalized Binet formulas is obtained. This formula is then compared with existing ones. Furthermore, by imposing specific constraints on the initial terms and coefficients of the recurrence relation, the formula is specialized to the Fibonacci sequence and other Fibonacci-like sequences by appropriately selecting $p_j, V_j = a_j$.

Key Words: Linear Recurrence Relations, Generalized tribonacci sequence, Unified explicit Binet formula.
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1. Introduction

A recursive relation is a method of defining a sequence in terms of itself. Generating of sequences numbers by recurrence relations have been the focus of several books (see, [1, 2, 3, 4]). Numbers sequences originated from the 3^{rd} order recurence relations (tribonacci numbers sequences) can be generalized in various means like changing the initial terms or the constant coefficients or both. Well-known generalizations are accomplished by appending the terms in the linear recurrence relation to generate the next term of the sequence [5, 6].

A generalized 3^{rd} order recurence relations (Tribonacci number sequence), $\{V_n\}$, is result of the recurrence relations with arbitrary constant coefficients and initial terms. Tribonacci sequence, first time mentioned in [7], therafter various generalizations of this sequence have appeared in the literature ([8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]). A novel graph labeling technique is proposed, utilizing three distinct number sequences: Tribonacci, Fibonacci, and Triangular is studied by [26]. Authors[27] Iidentify the Tribonacci numbers that are expressible as the product of two Fibonacci numbers. These generalization are also referred as the 3-step or 3^{rd}-order Fibonacci sequences. In order to determine the n^{th} tribonacci number $\{V_n\}$ rapidly and precisely, when n is large enough; mathematician have derived an explicit and quite useful formula to find the n^{th} term of the tribonacci numbers sequence, through the roots of the algebraic cubic equation. Sequences $\{V_n\}$ given by a generalized linear recurrence relation $V_n\left(a_j, p_j\right) = \sum_{j=1}^{3} p_j V_{n-j}$, $p_3 \neq 0$., For $j = 1, 2, 3$, $V_j = a_j$, p_j, initial terms and constant coefficients are any real numbers. Computing any term of the sequence when $n \geq 4$ requires knowing all preceding terms, which can be a computationally challenging task. So, some substitute formula, which provides any term of a sequence from the recursive relation in the form of index of the term, is preferred. Binet's formulas for the for tribonacci sequence sequences are known for fixed initial terms as for $a_j = 00, 1$ and constant coefficients $p_j = 1$. In the literature, varous forms of formulas are available in the general form and studied by [19, 20, 21] and [22, 23].

In this paper, homogeneous 3^{rd}order generalized linear recurrence relation, whose initial terms and constant coefficients are all considered in the most generalized form. Some starting terms of the sequences in the general form are worked out, then the generating function in the generalized form is obtained . An explicit, generalized analogous Binet's formulas are established, which are not only employing the roots of the algebraic cubic equation but has dependence on the initail terms as well as on the constant coefficients. Existing Binet-like formulas, previously derived for third-order recurrence relations and specifically for sequences such as Fibonacci, Lucas numbers, and others, are special cases of the obtained generalized formulas.

2. Preliminaries

Definition 1. We define the Generalized $3^{rd}-$ order sequence $\{V_n\}$ by the following linear recurrence relation:

$$V_n\left(a_j, p_j\right) = \sum_{j=1}^{3} p_j V_{n-j}, n \geq 4,$$

with initial terms $a_j = V_j$, $a_j, p_j,$ $j = 1, 2, 3,$ $p_3 \neq 0$ are arbitrary non–zero real numbers.

Definition 2. We define the Generalized Tribonacci Sequence $\{V_n\}$ by the following linear recurrence relation:

$$V_n(a_1, a_2, a_3, p_1, p_2, p_3) = p_1 V_{n-1} + p_2 F_{n-2} + p_3 V_{n-3},\ n \geq 4, \tag{1}$$

with the initial conditions, $a_j = V_j$, p_j, $j = 1, 2, 3$ are any non–zero real numbers.

The expression for $\{V_n\}$ in (1) is holds true, [2] for every integer $n \geq 4$.

2.1. Terms of the Generalized Tribonacci Sequence

The first few terms of the sequence$\{V_n\}$ defined in relation (1) are:

$$\{V_n\} = \left\{ \begin{array}{l} a_1, a_2, a_3, p_1 a_3 + p_2 a_2 + p_3 a_1, \left(p_1^2 + p_2\right) a_3 + \left(p_1 p_2 + p_3\right) a_2 + p_1 p_3 a_1, \\ \left(p_1^3 + p_3 + 2 p_1 p_2\right) a_3 + \left(p_1^2 p_2 + p_2^2 + p_1 p_3\right) a_2 + \left(p_1^2 p_3 + p_2 p_3\right) a_1 + \cdots \end{array} \right\}.$$

3. Generating Function

Theorem 1. *If $V(x)$ is generating function of the sequence (1) defined by the recursion*

$$V_n\left(a_j, p_j\right) = \sum_{j=1}^{k} p_j V_{n-j}, n \geq 4,$$

then

$$V(x) = \frac{f(x)}{1 - p_1 x - p_2 x^2 - p_3 x^3}, \tag{2}$$

where

$$f(x) = A_1 + A_2 x + A_3 x^2,$$

$$A_1 = V_1,\ A_2 = V_2 - p_1 V_1,\ A_3 = V_3 - p_1 V_2 - p_2 V_1.$$

Proof. Since

$$V(x) = \sum_{n=0}^{\infty} V_n x^n, \tag{3}$$

$$p_j x^j V(x) = \ p_j \sum_{n=j}^{\infty} V_{n-j} x^j, \ j = 1, 2, 3. \tag{4}$$

Now employing equations (2)-(4), we obtain generating function

$$V(x) \left[1 - p_1 x - p_2 x^2 - p_3 x^3\right] = f(x).$$

Therefore generating function has the rational form:

$$V(x) = \frac{A_1 + A_2 x + A_3 x^2}{1 - p_1 x - p_2 x^2},$$

Hence $V(x)$ is the generating function of (2) of the sequence $\{V_n\}$. ☐

4. Main Results

Theorem 2. *For the 3^{rd} order generalized linear recurrence relation (1), the n^{th} generalized number is*

$$V_n = \sum_{j=1}^{3} \left[\frac{\left[\sum\limits_{m=1}^{3} A_m \alpha_j^{3-m}\right] \left(\alpha_j^4 - {\alpha_j}^3\right)}{\left(1 + p_1\right) \alpha_j^3 + \sum\limits_{m=2}^{3} m \left(p_m - p_{m-1}\right) \alpha_j^{3-(m-1)} - 4p_4} \right] \alpha_j^{n-1}.$$

Proof. The generating function (2) of linear recurrence relation (1), can be expressed as:

$$V(x) = \frac{\sum\limits_{j=0}^{2} A_j x^j}{1 - \sum\limits_{j=1}^{3} p_j x^j} = \frac{\sum\limits_{j=0}^{2} A_j x^j}{h_3(x)}, \tag{5}$$

where $h_3(x) = 1 - \sum\limits_{j=1}^{3} p_j x^j$. Since α_j are roots of $x^3 - \sum\limits_{j=1}^{3} p_j x^{3-j} = 0$. So $\frac{1}{\alpha_j}$ are rootsof $1 - \sum\limits_{j=1}^{3} p_j x^j = 0$.

Miller [24] shown that the zeros of $h_3\left(\frac{1}{x}\right)$ are simple , consequently the zeros of $h_3(x)$are simple. Therefore

$$V(x) = \frac{\sum\limits_{j=0}^{2} A_j x^j}{\prod\limits_{j=1}^{3} (1 - \alpha_i x)}, \tag{6}$$

expressing right-hand side of (6) by using partial fraction decomposition, we have

$$V(x) = \sum_{j=1}^{3} \frac{R_j}{(1 - \alpha_i x)}, \tag{7}$$

where

$$R_j = \frac{\left(A_1 \alpha_j^2 + A_2 \alpha_j + A_3\right)}{\prod\limits_{\substack{m=1 \\ m \neq j}}^{3} (\alpha_j - \alpha_m)}.$$

Further,

$$R_j = \frac{\left(A_1\alpha_j + A_2 + \frac{A_3}{\alpha_j}\right)}{\prod\limits_{\substack{m=1\\ m\neq j}}^{3}\left(1-\alpha_m\left(\frac{1}{\alpha_j}\right)\right)} = \frac{\left(A_1\alpha_j + A_2 + \frac{A_3}{\alpha_j}\right)(-\alpha_j)}{h_3{}'\left(\frac{1}{\alpha_j}\right)}.$$

Now (7) become

$$V(x) = \frac{\sum\limits_{j=1}^{3} A_j x^j}{\prod\limits_{j=1}^{3}(1-\alpha_i x)} = \sum_{j=1}^{3}\frac{\left(A_1 + A_2 + \frac{A_3}{\alpha_j}\right)(-\alpha_j)}{h_3{}'\left(\frac{1}{\alpha_j}\right)}\frac{1}{(1-\alpha_i x)},$$

This implies

$$V(x) = \sum_{i=0}^{\infty}\left[\sum_{j=1}^{3}\frac{\left(A_1\alpha_j + A_2 + \frac{A_3}{\alpha_j}\right)(-\alpha_j)}{h_3{}'\left(\frac{1}{\alpha_j}\right)}\alpha_j^i x^i\right].$$

We have

$$V_n = \sum_{j=1}^{3}\frac{\left(A_1\alpha_j + A_2 + \frac{A_3}{\alpha_j}\right)(-\alpha_j)(\alpha_j)^n}{h_k{}'\left(\frac{1}{\alpha_j}\right)}.$$

Since

$$h_3{}'\left(\frac{1}{\alpha_j}\right) = -\sum_{m=1}^{3} m p_m\left(\frac{1}{\alpha_j}\right)^{m-1}, 1-\sum_{j=1}^{3} p_j\left(\frac{1}{\alpha_j}\right)^{m-1} = 0,\ 1\le j\le 3.$$

Therefore

$$V_n = \sum_{j=1}^{3}\frac{\left(A_1\alpha_j + \frac{A_3}{\alpha_j}\right)(\alpha_j)(\alpha_j)^n\left(1-\frac{1}{\alpha_j}\right)}{(1+p_1)+2(p_2-p_1)\left(\frac{1}{\alpha_j}\right)+3(p_3-p_2)\left(\frac{1}{\alpha_j}\right)^2-4p_3\left(\frac{1}{\alpha_j}\right)^3}. \tag{8}$$

On further simplification we have

$$V_n = \sum_{j=1}^{3}\frac{\left(A_1\alpha_j^2 + A_2\alpha_j + A_3\right)\left(\alpha_j^4 - \alpha_j{}^3\right)}{(1+p_1)\alpha_j^3 + 2(p_2-p_1)\alpha_j^2 + 3(p_3-p_2)\alpha_j - 4p_3}\alpha_j^{n-1}.$$

$$V_n = \sum_{j=1}^{3}\left[\frac{\sum\limits_{m=1}^{3} A_m\alpha_j^{3-m}\left(\alpha_j^4-\alpha_j{}^3\right)}{(1+p_1)\alpha_j^3 + \sum\limits_{m=2}^{3} m(p_m - p_{m-1})\alpha_j^{3-(m-1)} - 4p_3}\right]\alpha_j^{n-1}. \tag{9}$$

Hence equation (9) is most generalized Binet type formula for the 3^{th} order generalized recurrence relations. □

Corollary 1. When $p_1 = p_2 = p_3 = 1$ and $V_1 = 0,\ V_2 = 0,\ V_3 = 1$ implies that $A_1 = 0,\ A_2 = 0,\ A_3 = 1$, then from generalized Binet formula (9), we obtain

$$V_n = \sum_{j=0}^{3}\frac{\left(\alpha_j^4 - \alpha_j{}^3\right)}{2\alpha_j^3 - 4}\alpha_j^n, \tag{10}$$

which is the result obtained by Spickerman and Joyner [6].

Corollary 2. For the 2^{nd} order recurence relation. the generalized Binet formula (9) reduces to ,

$$V_n = \sum_{j=1}^{2} \frac{(A_1\alpha_j + A_2)\left(\alpha_j^3 - {\alpha_j}^2\right)}{(1+p_1)\,\alpha_j^2 + 2\,(p_2 - p_1)\,\alpha_j - 3p_2}\alpha_j^{n-1}. \tag{11}$$

where α_j are roots of the equation $x^2 - x - 1 = 0$. Equation (11) deduced from the generalization is the result obtained by *Spickerman and Joyner* [6].

Corollary 3. Binet formula for 2^{nd} order recurence relation.
If $p_i = 1$, for all i, $V_1 = 0$ and $V_2 = 1$, then from the generalized Binet formula (9), reduces to

$$V_n = \sum_{j=1}^{2} \frac{\left(\alpha_j^3 - {\alpha_j}^2\right)}{2\alpha_j^2 - 3}\alpha_j^{n}, \tag{12}$$

equation (12) deduced from the generalization is the result, obtained by *Spickerman and Joyner* [6].

Corollary 4. When $p_1 = p_2 = 1$ and $V_1 = 0$, $V_2 = 1$ implies $A_1 = 0$, $A_2 = 1$, then from generalized Binet formula (9) , we obtain

$$V_n = \frac{(\alpha - 1)}{2\alpha - 3}\alpha^{n+1} + \frac{(\beta - 1)}{2\beta - 3}\beta^{n+1}, \tag{13}$$

where α and β are roots of the equation $x^2 - x - 1 = 0$.

Theorem 3. *The generalized Binet formul for V_n obtained in Theorem (2) can be expressed as*

$$V_n = \sum_{j=1}^{3} \left[\frac{\left(\sum\limits_{m=1}^{3} A_m \alpha_j^{3-m}\right)(\alpha_j - 1)}{\begin{array}{c}(1+p_1) + \sum\limits_{m=2}^{3} m\,(p_m - p_{m-1})\,\alpha_j^{(m-1)} \\ + 4\left[\alpha_j - (p_1 + 1) - \sum\limits_{m=2}^{3} (p_m - p_{m-1})\,{\alpha_j}^{-(m-1)}\right]\end{array}} \right] \alpha_j^{n-1}. \tag{14}$$

Proof. Since we have the result from Theorem (2)

$$V_n = \sum_{j=1}^{3} \left[\frac{\sum\limits_{m=1}^{3} A_m \alpha_j^{3-m}\left(\alpha_j^4 - {\alpha_j}^3\right)}{(1+p_1)\,\alpha_j^3 + \sum\limits_{m=2}^{3} m\,(p_m - p_{m-1})\,\alpha_j^{3-(m-1)} - 4p_3} \right] \alpha_j^{n-1}. \tag{15}$$

Now the from characteristic equation $x^3 - \sum\limits_{j=1}^{3} p_j x^{3-j} = 0$ of the recurrence relation we have $\alpha_j^3 - \sum\limits_{j=1}^{3} p_j \alpha_j^{3-j} = 0$, where α_j, $(j = 1,2,3)$ are roots of the characteristic equation.

$$\left(\alpha_j^3 - \sum_{j=1}^{3} p_j \alpha_j^{3-j}\right)(1 - \alpha_j) = 0.$$

We obtain

$$\alpha_j{}^3\left[\alpha_j-(p_1+1)-\sum_{m=2}^{3}\left(p_m-p_{m-1}\right)\alpha_j{}^{-(m-1)}\right]=-p_3.$$

This equation can be expressed as

$$\left[\alpha_j-(p_1+1)-\sum_{m=2}^{3}\left(p_m-p_{m-1}\right)\alpha_j{}^{-(m-1)}\right]=-p_3\alpha_j{}^{-3}. \tag{16}$$

From equations (15)- (16), we obtain

$$V_n=\sum_{j=1}^{3}\left[\frac{\left(A_1\alpha_j^2+A_2\alpha_j+A_3\right)\left(\alpha_j-1\right)}{(1+p_1)+2\left(p_2-p_2\right)\alpha_j+3\left[\alpha_j-(p_1+1)-2\left(p_3-p_2\right)\alpha_j{}^{-1}\right]}\right]\alpha_j^{n-1}. \tag{17}$$

which is another form of generalized Binet formula for generalized 3^{rd} order recurrence relations. Hence the Theorem. □

Corollary 5. When $p_1=p_2=p_3=1$ and $V_1=0,\ V_2=0,\ V_3=1$ implies $A_1=0,\ A_2=0,\ A_3=1$, then from generalized Binet formula (17) , we obtain

$$V_n=\sum_{j=1}^{3}\left[\frac{(\alpha_j-1)}{2+4\left(\alpha_j-2\right)}\right]\alpha_j^{n-1}. \tag{18}$$

which is the same result obtained by *Dresden and Du* [5].

Corollary 6. For the 2^{nd} order recurrsive relation the generalized Binet formula obtained in Theorem (3) equation (17) become

$$V_n=\sum_{j=1}^{2}\left[\frac{\left(\sum_{m=1}^{2}A_m\alpha_j^{k-m}\right)\left(\alpha_j-1\right)}{(1+p_1)+2\left(p_2-p_1\right)\alpha_j+3\left[\alpha_j-(p_1+1)-\left(p_2-p_1\right)\alpha^{-1}\right]}\right]\alpha_j^{n-1}. \tag{19}$$

α_j are roots of the $x^2-p_1x-p_2=0$.

Corollary 7. For the 2^{nd} order recurrsive relation the with $p_i=1$, for all i , $V_1=0$, $V_2=1$ generalized Binet formula obtained in Theorem (3) equation (17) become

$$V_n=\sum_{j=1}^{2}\left[\frac{(\alpha_j-1)}{2+3\left[\alpha_j-2\right]}\right]\alpha_j^{n-1}. \tag{20}$$

α_j are roots of the $x^2-x-1=0$.

5. Special Cases

Remark 1. With initial conditions $V_0=0, V_1=1,\ V_2=1$, and $p_1=p_2=p_3=1$, recurrence relation (1) is known as the tribonacci sequences is also known as the generalized Lucas sequence and is denoted by T_n [14]. The first few terms of the sequence deduced from the above generalization are:

$$\{V_n\}_{n\geq 0}=\{T_n\}=\{0,1,1,2,4,7,13,24,44,81,149,274,504,927,1705,3136,\cdots\}.$$

Remark 2. If we substitute the initial conditions $V_0=3, V_1=1,\ V_2=3$,and $p_1=p_2=p_3=1$ in (1) , it reduces to sequence K_n sequence which is explained in [25].The first few terms of the K_n are:

$$\{V_n\}_{n\geq 0}=\{K_n\}=\{3,1,3,7,11,21,39,71,131,241,443,815,1499,2757,5071,9327,\cdots\}.$$

6. Discussion and Conclusion

The 3^{rd}-order generalized recurrence relation is considered, where both the initial terms and constant coefficients are chosen arbitrarily. Subsequently, the generating function in its general form is derived. Employing this generating function, explicit Binet-like formulas are obtained in terms of the roots of the characteristic equation, initial terms, and constant coefficients. It is demonstrated that many existing results studied by various authors become special cases of the derived formulas. In the future, the third-order generalized sequence can be explored more thoroughly, and the study may be extended using these generalized formulas. It would be interesting to investigate alternative approaches, such as matrices, combinatorial arguments, or number theory, to uncover additional identities and theorems based on the new explicit generalized results.

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